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220

Joseph A. Ball
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Editors

Recent Progress in Operator Theory and Its Applications

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Operator Theory: Advances and Applications

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Introduction

This volume contains the Proceedings of the Twentieth International Workshop on Operator Theory and Applications (IWOTA), held at Hotel Real de Minas in Guanajuato, Mexico, during September 21–25, 2009. This was the twentieth IWOTA; in fact, the workshop was held biannually since 1981, and annually in the recent years (starting 2002) rotating among eleven countries on three continents. Previous IWOTA meetings were held at:

Santa Monica, CA, USA (1981)
J.W. Helton, Chair

Rehovot, Israel (1983) – *Oper. Theory Adv. Appl.* 12;
H. Dym and I. Gohberg, Co-chairs

Amsterdam, Netherlands (1985) – *Oper. Theory Adv. Appl.* 19;
M.A. Kaashoek, Chair

Mesa, AZ, USA (1987) – *Oper. Theory Adv. Appl.* 35;
J.W. Helton and L. Rodman, Co-chairs

Rotterdam, Netherlands (1989) – *Oper. Theory Adv. Appl.* 50;
H. Bart, Chair

Sapporo, Japan (1991) – *Oper. Theory Adv. Appl.* 59;
T. Ando, Chair

Vienna, Austria (1993) – *Oper. Theory Adv. Appl.* 80;
H. Langer, Chair

Regensburg, Germany (1995) – *Oper. Theory Adv. Appl.* 102 and 103;
R. Mennicken, Chair

Bloomington, IN, USA (1996) – *Oper. Theory Adv. Appl.* 115;
H. Bercovici and C. Foiaş, Co-chairs

Groningen, Netherlands, (1998) – *Oper. Theory Adv. Appl.* 124;
A. Dijksma, Chair

Bordeaux, France (2000) – *Oper. Theory Adv. Appl.* 129;
N. Nikolskii, Chair

Faro, Portugal (2000) – *Oper. Theory Adv. Appl.* 142;
A.F. Dos Santos and N. Manojlovic, Co-chairs

Blacksburg, VA, USA (2002) – *Oper. Theory Adv. Appl.* 149;
J. Ball, Chair

Cagliari, Italy (2003) – *Oper. Theory Adv. Appl.* 160;
S. Seatzy and C. van der Mee, Co-chairs

Newcastle, UK (2004) – *Oper. Theory Adv. Appl.* 171;
M.A. Dritshel and N. Young, Co-chairs

Storrs, CT, USA (2005) – *Oper. Theory Adv. Appl.* 179;
V. Olshevsky, Chair

Seoul, Korea (2006) – *Oper. Theory Adv. Appl.* 187;
Woo Young Lee, Chair

Potchefstroom, South Africa (2007) – *Oper. Theory Adv. Appl.* 195;
K. Grobler and G. Groenewald, Co-chairs

Williamsburg, VA, USA, (2008) – *Oper. Theory Adv. Appl.* 202 and 203;
L. Rodman, Chair

Guanajuato, Mexico (2009) – *Oper. Theory Adv. Appl.* 220;
N. Vasilevski, Chair

Berlin, Germany (2010) – *Oper. Theory Adv. Appl.* 221;
J. Behrndt, K.-H. Förster and C. Trunk, Co-chairs

Seville, Spain (2011)
A. Montes Rodríguez, Chair.

Consistent with the topics of recent IWOTA meetings, IWOTA 2009 was designed as a comprehensive, inclusive conference covering all aspects of theoretical and applied operator theory, ranging from classical analysis, differential and integral equations, complex and harmonic analysis to mathematical physics, mathematical systems and control theory, signal processing and numerical analysis. The conference brought together international experts for a week-long stay at Hotel Real de Minas, in an atmosphere conducive to fruitful professional interactions. These Proceedings reflect the high quality of the papers presented at the conference. In addition to fourteen plenary sessions, IWOTA 2009 included the following special sessions:

Bergman and Segal-Bargmann spaces and Toeplitz operators

Factorization problems, Wiener-Hopf and Fredholm operators

Hypercomplex operator theory

Indefinite inner product spaces and spectral problems

Multivariable operator theory

Operators on function spaces

Pseudodifferential operators and related topics

Solution techniques for partial differential equations

Spectral theory and its applications

Toeplitz/rank structured tensors and matrices.

This volume contains twenty-one solicited articles by speakers at the workshop, ranging from expository surveys to original research papers, each carefully refereed. All contributions reflect recent developments in operator theory and its applications.

The organizers gratefully acknowledge the support of the following institutions:

CONACYT (Consejo Nacional de Ciencia y Tecnología, Mexico)

Sociedad Matemática Mexicana (Mexico)

Department of Mathematics, CINVESTAV (Mexico)

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23 October 2011

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Advances and Applications, Vol. 220, 1–10

Exponential Decay of Semigroups for Second-order Non-selfadjoint Linear Differential Equations

Nikita Artamonov

Abstract. The Cauchy problem for second-order linear differential equation

$$u''(t) + Du'(t) + Au(t) = 0$$

in Hilbert space H with a sectorial operator A and an accretive operator D is studied. Sufficient conditions for exponential decay of the solutions are obtained.

Mathematics Subject Classification (2000). Primary 47D06, 34G10; Secondary 47B44, 35G15.

Keywords. Accretive operator, sectorial operator, C_0-semigroup, second-order linear differential equation, spectrum.

Many linearized equations of mechanics and mathematical physics can be reduced to a linear differential equation

$$u''(t) + Du'(t) + Au(t) = 0, \tag{0.1}$$

where $u(t)$ is a vector-valued function in an appropriate (finite- or infinite-dimensional) Hilbert space H, D and A are linear (bounded or unbounded) operators on H. Properties of the differential equation (0.1) are closely connected with spectral properties of a quadric pencil

$$L(\lambda) = \lambda^2 + \lambda D + A, \quad \lambda \in \mathbb{C}$$

which is obtained by substituting exponential functions $u(t) = \exp(\lambda t)x$, $x \in H$ into (0.1). In many applications A is a self-adjoint positive definite operator, D is a self-adjoint positive definite or an accretive operator (see definition in Section 1). In this case the differential equation (0.1) and spectral properties of the related quadric pencil $L(\lambda)$ are well studied, see [2, 6, 7, 8, 10, 11, 12, 13, 15] and references therein. It was obtained a localization of the pencil's spectrum, sufficient

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conditions of the completeness of eigen- and adjoint vectors of the pencil $L(\lambda)$ and it was proved, that all solutions of (0.1) exponentially decay. The exponential decay means, that the total energy exponentially decreases and corresponding mechanical system is stable. In paper [16] was studied spectral properties of the pencil $L(\lambda)$ for a self-adjoint non-positive definite operator A and an accretive operator D.

But some models of continuous mechanics are reduced to differential equation (0.1) with sectorial operator A, see [1, 9, 17] and references therein. In this cases methods, developed for self-adjoint operator A, cannot be applied.

The aim of this paper is the study of a Cauchy problem for second-order linear differential equation (0.1) in a Hilbert space H with initial conditions

$$u(0) = u_0 \quad u'(0) = u_1. \tag{0.2}$$

The shiffness operator A is assumed to be a sectorial operator, the damping operator D is assumed to be an accretive operator.

By $\mathcal{L}(H', H'')$ denote a space of bounded operators acting from a Hilbert space H' to a Hilbert space H''. $\mathcal{L}(H) = \mathcal{L}(H, H)$ is an algebra of bounded operators acting on Hilbert space H.

1. Preliminary results

First let us recall some definitions [4, 14].

Definition 1.1. Linear operator B with dense domain $\mathcal{D}(B)$ is called *accretive* if $\operatorname{Re}(Bx, x) \geq 0$ for all $x \in \mathcal{D}(B)$ and *m-accretive*, if the range of operator $B + \omega I$ is dense in H for some $\omega > 0$.

An accretive operator B is m-accretive iff B has not accretive extensions [14]. For m-accretive operator

$$\rho(B) \supset \{\lambda \in \mathbb{C} \ : \ \operatorname{Re}\lambda < 0\}.$$

Definition 1.2. An accretive operator B is called *sectorial* or *ω-accretive* if for some $\omega \in [0, \pi/2)$

$$\big|\operatorname{Im}(Bx, x)\big| \leq \tan(\omega)\operatorname{Re}(Bx, x) \quad x \in \mathcal{D}(B).$$

If a sectorial operator has not sectorial extensions, then it is called *m-sectorial* or *m-ω-accretive.*

The sectorial property means that the numerical range of the operator B belongs to a sector

$$\{z \in \mathbb{C} \mid |\operatorname{Im} z| \leq \tan(\omega)\operatorname{Re} z\}.$$

For a sectorial operator B there exist [14] a self-adjoint non-negative operator T_B and a self-adjoint operator $S_B \in \mathcal{L}(H)$, $\|S_B\| \leq \tan(\omega)$ such that

$$\operatorname{Re}(Bx, x) = (T_B^{1/2}x, T_B^{1/2}x), \quad B \subset T_B^{1/2}(I + iS_B)T_B^{1/2}$$

and $B = T_B^{1/2}(I + iS_B)T_B^{1/2}$ iff B is m-sectorial.

Throughout this paper we will assume, that

(A) Operator $A : \mathcal{D}(A) \subset H \to H$ is m-sectorial and for some positive a_0

$$\mathrm{Re}(Ax, x) \geq a_0(x, x) \quad x \in \mathcal{D}(A).$$

Since A is m-sectorial there exist a self-adjoint positive definite operator T and a self-adjoint $S \in \mathcal{L}(H)$, such that

$$\mathrm{Re}(Ax, x) = (T^{1/2}x, T^{1/2}x) \geq a_0(x, x), \quad x \in \mathcal{D}(A), \quad A = T^{1/2}(I + iS)T^{1/2}.$$

The operator A is invertible and

$$A^{-1} = T^{-1/2}(I + iS)^{-1}T^{-1/2}.$$

By H_s ($s \in \mathbb{R}$) denote a collection of Hilbert spaces generated by a self-adjoint operator $T^{1/2}$:

- for $s \geq 0$ $H_s = \mathcal{D}(T^{s/2})$ endowed with a norm $\|x\|_s = \|T^{s/2}x\|$;
- for $s < 0$ H_s is a closure of H with respect to the norm $\|\cdot\|_s$.

Obviously $H_0 = H$. The operator $T^{1/2}$ can be considered now as a unitary operator mapping H_s on H_{s-1}. A is a bounded operator $A \in \mathcal{L}(H_2, H_0)$ and it can be extended to a bounded operator $\tilde{A} \in \mathcal{L}(H_1, H_{-1})$. The inverse operator A^{-1} can be extended to a bounded operator $\tilde{A}^{-1} \in \mathcal{L}(H_{-1}, H_1)$.

By $(\cdot, \cdot)_{-1,1}$ denote a duality pairing on $H_{-1} \times H_1$. Note, that for all $x \in H_{-1}$ and $y \in H_1$ we have

$$\left|(x, y)_{-1,1}\right| \leq \|x\|_{-1} \cdot \|y\|_1$$

and $(x, y)_{-1,1} = (x, y)$ if $x \in H$. Further,

$$\mathrm{Re}(\tilde{A}x, x)_{-1,1} = (Tx, x)_{-1,1} = (T^{1/2}x, T^{1/2}x) = \|x\|_1^2, \quad x \in H_1 = \mathcal{D}(T^{1/2}).$$

Denote $\tilde{S} = T^{1/2}ST^{1/2} \in \mathcal{L}(H_1, H_{-1})$. Then, for the operator $\tilde{A}$ we have a representation $\tilde{A} = T + i\tilde{S}$ and

$$\mathrm{Im}(\tilde{A}x, x)_{-1,1} = (\tilde{S}x, x)_{-1,1} \quad x \in H_1.$$

Also $(\tilde{S}x, y)_{-1,1} = \overline{(\tilde{S}y, x)}_{-1,1}$ for all $x, y \in H_1$.

Following paper [11] we will assume

(B) D is a bounded operator $D \in \mathcal{L}(H_1, H_{-1})$, and

$$\beta = \inf_{x \in H_1, x \neq 0} \frac{\mathrm{Re}(Dx, x)_{-1,1}}{\|x\|^2} > 0. \tag{1.1}$$

Operator $T^{-1/2}$ is a unitary operator mapping H_s on H_{s+1}, therefore an operator $D' = T^{-1/2}DT^{-1/2}$, acting on H, is bounded. Let

$$D_1 = \frac{1}{2}T^{1/2}\Big(D' + (D')^*\Big)T^{1/2} \quad D_2 = \frac{1}{2i}T^{1/2}\Big(D' - (D')^*\Big)T^{1/2},$$

Obviously $D_1, D_2 \in \mathcal{L}(H_1, H_{-1})$, $D = D_1 + iD_2$ and for all $x \in H_1$

$$\mathrm{Re}(Dx, x)_{-1,1} = (D_1x, x)_{-1,1} \geq \beta\|x\|^2, \quad \mathrm{Im}(Dx, x)_{-1,1} = (D_2x, x)_{-1,1}.$$

Also $(D_jx, y)_{-1,1} = \overline{(D_jy, x)}_{-1,1}$ for all $x, y \in H_1$ $(j = 1, 2)$.

2. Main result

Definition 2.1. A vector-valued function $u(t) \in H_1$ is called a solution of the differential equation (0.1) if $u'(t) \in H_1$, $u''(t) \in H$, $Du'(t) + \tilde{A}u(t) \in H$ and

$$u''(t) + Du'(t) + \tilde{A}u(t) = 0 \tag{2.1}$$

If $u(t)$ is a solution of (2.1), then a vector-function

$$\mathbf{x}(t) = \begin{pmatrix} u'(t) \\ u(t) \end{pmatrix}$$

(formally) satisfies a first-order differential equation

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) \tag{2.2}$$

with a block operator matrix

$$\mathbf{A} = \begin{pmatrix} -D & -\tilde{A} \\ I & 0 \end{pmatrix}.$$

From mechanical viewpoint it is most natural to consider the equation (2.2) in an "energy" space $\mathcal{H} = H \times H_1$ with a dense domain of the operator $\mathbf{A}$ [6, 7, 11, 16]

$$\mathcal{D}(\mathbf{A}) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \, x_1, x_2 \in H_1, \ Dx_1 + \tilde{A}x_2 \in H \right\}.$$

An inverse of $\mathbf{A}$ is formally defined by a block operator matrix

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & I \\ -\tilde{A}^{-1} & -\tilde{A}^{-1}D \end{pmatrix}.$$

Let $\mathbf{y} = (y_1, y_2)^\top \in \mathcal{H} = H \times H_1$, then

$$\mathbf{A}^{-1}\mathbf{y} = \begin{pmatrix} y_2 \\ -\tilde{A}^{-1}y_1 - \tilde{A}^{-1}Dy_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Since $\tilde{A}^{-1} \in \mathcal{L}(H_{-1}, H_1)$ and $D \in \mathcal{L}(H_1, H_{-1})$, then $\tilde{A}^{-1}D \in \mathcal{L}(H_1, H_1)$. Therefore $-\tilde{A}^{-1}y_1 - \tilde{A}^{-1}Dy_2 \in H_1$ and $\mathbf{A}^{-1}\mathbf{y} \in H_1 \times H_1$. Moreover,

$$Dx_1 + \tilde{A}x_2 = Dy_2 + \tilde{A}\left(-\tilde{A}^{-1}y_1 - \tilde{A}^{-1}Dy_2\right) = -y_1 \in H.$$

Thus $\mathbf{A}^{-1}\mathbf{y} \in \mathcal{D}(\mathbf{A})$. Since $I \in \mathcal{L}(H_1, H)$ the operator $\mathbf{A}^{-1}$ is bounded and therefore the operator $\mathbf{A}$ is closed and $0 \in \rho(\mathbf{A})$.

Let $(\mathbf{x}, \mathbf{y})_{\mathcal{H}}$ be a natural scalar product on $\mathcal{H} = H \times H_1$ and $\|\mathbf{x}\|^2_{\mathcal{H}} = (\mathbf{x}, \mathbf{y})_{\mathcal{H}}$.

If operator A is self-adjoint, the spectral properties of operator $\mathbf{A}$ are well studied: $-\mathbf{A}$ is an m-accretive operator in the Hilbert space $\mathcal{H} = H \times H_1$ (see [2, 6, 7, 8, 10, 11] and references therein) and, consequently, $\mathbf{A}$ is a generator of a C_0-semigroup. Thus, differential equation (2.2) (and equation (2.1)) is correctly solvable in the space $\mathcal{H}$ for all $\mathbf{x}(0) = (u_1, u_0)^\top \in \mathcal{D}(\mathbf{A})$. Moreover, in this case operator $\mathbf{A}$ is a generator of a contraction semigroup [7]. It implies, that all solutions of (2.2) (and (2.1)) exponentially decay, i.e., for some $C, \omega > 0$

$$\|\mathbf{x}(t)\|_{\mathcal{H}} \le C \exp(-\omega t)\|\mathbf{x}(0)\|_{\mathcal{H}} \quad t \ge 0.$$

For non-selfadjoint A operator $(-\mathbf{A})$ is not longer accretive in the space $\mathcal{H}$ with respect to the standard scalar product. But, under some assumptions, one can define a new scalar product on $\mathcal{H}$, which is topologically equivalent to the given one, such that an operator $(-\mathbf{A}-qI)$ (for some $q \geq 0$) is m-accretive and therefore the operator $\mathbf{A}$ is a generator of a C_0-semigroup on $\mathcal{H}$. If $q > 0$, then $\mathbf{A}$ is a generator of a contraction semigroup and all solutions of (2.2) exponentially decay.

Let $k \in (0, \beta)$ (β is defined by (1.1)). Consider on the space $\mathcal{H}$ a sesquilinear form

$$[\mathbf{x}, \mathbf{y}]_{\mathcal{H}} = (T^{1/2}x_2, T^{1/2}y_2) + k(D_1x_2, y_2)_{-1,1} - k^2(x_2, y_2) + (x_1 + kx_2, y_1 + ky_2),$$
$$\mathbf{x} = (x_1, x_2)^\top,\ \mathbf{y} = (y_1, y_2)^\top \in \mathcal{H}.$$

Obviously, $[\mathbf{x}, \mathbf{y}] = \overline{[\mathbf{y}, \mathbf{x}]}$ and

$$[\mathbf{x}, \mathbf{x}]_{\mathcal{H}} = \|x_2\|_1^2 + k(D_1x_2, x_2)_{-1,1} + \|x_1\|^2 + 2k\operatorname{Re}(x_1, x_2).$$

Since $(D_1x, x)_{-1,1} = \operatorname{Re}(Dx, x)_{-1,1} \geq \beta\|x\|^2$ and

$$2|\operatorname{Re}(x_1, x_2)| \leq 2|(x_1, x_2)| \leq 2\|x_1\| \cdot \|x_2\| \leq \frac{\|x_1\|^2}{\beta} + \beta\|x_2\|^2,$$

then

$$[\mathbf{x}, \mathbf{x}]_{\mathcal{H}} \geq \|x_2\|_1^2 + k\Big((D_1x, x)_{-1,1} - \beta\|x_2\|^2\Big) + \left(1 - \frac{k}{\beta}\right)\|x_1\|^2$$
$$\geq \|x_2\|_1^2 + \left(1 - \frac{k}{\beta}\right)\|x_1\|^2.$$

Inequalities[1] $|(D_1x, x)_{-1,1}| \leq \|D_1x\|_{-1} \cdot \|x\|_1 \leq \|D_1\| \cdot \|x\|_1^2$ and $\|x\|_1^2 \geq a_0\|x\|^2$ imply

$$[\mathbf{x}, \mathbf{x}]_{\mathcal{H}} \leq \Big(1 + k\|D_1\|\Big)\|x_2\|_1^2 + k\beta\|x_2\|^2 + \left(1 + \frac{k}{\beta}\right)\|x_1\|^2$$
$$\leq \left(1 + k\|D_1\| + \frac{k\beta}{a_0}\right)\|x_2\|_1^2 + \left(1 + \frac{k}{\beta}\right)\|x_1\|^2.$$

Thus,

$$\left(1 - \frac{k}{\beta}\right)\|\mathbf{x}\|_{\mathcal{H}}^2 \leq [\mathbf{x}, \mathbf{x}]_{\mathcal{H}} \leq \text{const}\,\|\mathbf{x}\|_{\mathcal{H}}^2$$

and $[\cdot, \cdot]_{\mathcal{H}}$ is a scalar product on $\mathcal{H}$, which is topologically equivalent to the given one. Denote $|\mathbf{x}|_{\mathcal{H}}^2 = [\mathbf{x}, \mathbf{x}]_{\mathcal{H}}$.

Theorem 2.2. *Let the assumptions* (A) *and* (B) *hold and for some* $k \in (0, \beta)$ *and* $m \in (0, 1]$

$$\omega_1 = \inf_{x \in H_1, x \neq 0} \frac{\frac{1}{k}(D_1x, x)_{-1,1} - \|x\|^2 - \frac{1}{4m}\big\|(\frac{1}{k}\tilde{S} - D_2)x\big\|_{-1}}{\|x\|^2} \geq 0. \tag{2.3}$$

[1] $\|D_1\|$ is a norm of operator $D_1 \in \mathcal{L}(H_1, H_{-1})$, i.e., $\|D_1\| = \sup_{x \in H_1, x \neq 0} \|D_1x\|_{-1}/\|x\|_1$.

Then the operator $\mathbf{A}$ *is a generator of a* C_0*-semigroup* $\mathcal{T}(t) = \exp\{t\mathbf{A}\}$ $(t \geq 0)$ *and*

$$\left\|\mathcal{T}(t)\right\|_{\mathcal{H}} \leq \mathrm{const} \cdot \exp(-tk\theta)$$

where

$$\theta = \min\left\{\frac{\omega_1}{2}, \frac{1-m}{\omega_2}\right\} \geq 0$$

and[2]

$$\omega_2 = \sup_{x \in H_1, x \neq 0} \frac{\|x\|_1^2 + k(D_1 x, x)_{-1,1} + k^2\|x\|^2}{\|x\|_1^2} \tag{2.4}$$

Proof. For $\mathbf{x} = (x_1, x_2)^\top \in \mathcal{D}(\mathbf{A})$ let us consider a quadric form

$$\begin{aligned}
[\mathbf{A}\mathbf{x}, \mathbf{x}]_{\mathcal{H}} &= (T^{1/2}x_1, T^{1/2}x_2) + k(D_1x_1, x_2)_{-1,1} - k^2(x_1, x_2) \\
&\quad + (-Dx_1 - \tilde{A}x_2 + kx_1, x_1 + kx_2) \\
&= (Tx_1, x_2)_{-1,1} + k(D_1x_1, x_2)_{-1,1} - (Dx_1, x_1)_{-1,1} \\
&\quad - (\tilde{A}x_2, x_1)_{-1,1} + k(x_1, x_1) - k(Dx_1, x_2)_{-1,1} - k(\tilde{A}x_2, x_2)_{-1,1} \\
&= -(Dx_1, x_1)_{-1,1} + k(x_1, x_1) - k(\tilde{A}x_2, x_2)_{-1,1} - ik(D_2x_1, x_2)_{-1,1} \\
&\quad + (Tx_1, x_2)_{-1,1} - (Tx_2, x_1)_{-1,1} - i(\tilde{S}x_2, x_1)_{-1,1}
\end{aligned}$$

We used decompositions $\tilde{A} = T + i\tilde{S}$ and $D = D_1 + iD_2$. Consequently,

$$\begin{aligned}
\mathrm{Re}[\mathbf{A}\mathbf{x}, \mathbf{x}]_{\mathcal{H}} &= -(D_1x_1, x_1)_{-1,1} + k(x_1, x_1) - k(Tx_2, x_2)_{-1,1} \\
&\quad - \mathrm{Re}\big(ik(D_2x_1, x_2)_{-1,1} + i(\tilde{S}x_2, x_1)_{-1,1}\big) \\
&= -(D_1x_1, x_1)_{-1,1} + k\|x_1\|^2 - k\|x_2\|_1^2 \\
&\quad - \mathrm{Im}\big((\tilde{S}x_1, x_2)_{-1,1} - k(D_2x_1, x_2)_{-1,1}\big)
\end{aligned}$$

and

$$-\frac{1}{k}\mathrm{Re}[\mathbf{A}\mathbf{x}, \mathbf{x}]_{\mathcal{H}} = \frac{1}{k}(D_1x_1, x_1)_{-1,1} - \|x_1\|^2 + \|x_2\|_1^2 + \mathrm{Im}\left(\left(\frac{1}{k}\tilde{S} - D_2\right)x_1, x_2\right)_{-1,1}.$$

Since

$$\begin{aligned}
\left|\left(\left(\frac{1}{k}\tilde{S} - D_2\right)x_1, x_2\right)_{-1,1}\right| &\leq \left\|\left(\frac{1}{k}\tilde{S} - D_2\right)x_1\right\|_{-1} \cdot \|x_2\|_1 \\
&\leq \frac{1}{4m}\left\|\left(\frac{1}{k}\tilde{S} - D_2\right)x_1\right\|_{-1}^2 + m\|x_2\|_1^2,
\end{aligned}$$

then

$$\begin{aligned}
-\frac{1}{k}\mathrm{Re}[\mathbf{A}\mathbf{x}, \mathbf{x}]_{\mathcal{H}} &\geq \frac{1}{k}(D_1x_1, x_1)_{-1,1} - \|x_1\|^2 - \frac{1}{4m}\left\|\left(\frac{1}{k}\tilde{S} - D_2\right)x_1\right\|_{-1}^2 + (1-m)\|x_2\|_1^2 \\
&\geq \omega_1\|x_1\|^2 + (1-m)\|x_2\|_1^2.
\end{aligned}$$

[2]Obviously, $\omega_2 \leq 1 + k\|D_1\| + k^2/a_0$.

Further, an inequality

$$2k|\operatorname{Re}(x_1, x_2)| \le 2|(x_1, kx_2)| \le 2\|x_1\| \cdot \|kx_2\| \le \|x_1\|^2 + k^2\|x_2\|^2$$

implies

$$[\mathbf{x}, \mathbf{x}]_{\mathcal{H}} \le 2\|x_1\|^2 + \|x_2\|_1^2 + k(D_1x_2, x_2)_{-1,1} + k^2\|x_2\|^2 \le 2\|x_1\|^2 + \omega_2\|x_2\|_1^2. \quad (2.5)$$

Thus

$$-\frac{1}{k}\operatorname{Re}[\mathbf{A}\mathbf{x}, \mathbf{x}]_{\mathcal{H}} \ge \omega_1\|x_1\|^2 + (1-m)\|x_2\|_1^2 \ge \theta(2\|x_1\|^2 + \omega_2\|x_2\|_1^2) \ge \theta[\mathbf{x}, \mathbf{x}]_{\mathcal{H}}$$

and an operator $(-\mathbf{A} - k\theta I)$ is accretive. Moreover, the operator $(-\mathbf{A} - k\theta I)$ is m-accretive (since $0 \in \rho(\mathbf{A})$) and[3]

$$\rho\left(-\mathbf{A} - k\theta I\right) \subset \{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda < 0\} \Rightarrow \rho(-\mathbf{A}) \supset \{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda < k\theta\}.$$

Therefore, the operator $\mathbf{A}$ is a generator of a C_0-semigroup [4, 5] $\mathcal{T}(t) = \exp\{t\mathbf{A}\}$, $t \ge 0$ and

$$\left|\mathcal{T}(t)\right|_{\mathcal{H}} \le \exp(-k\theta t), \quad t \ge 0.$$

On the space $\mathcal{H}$ norms $|\mathbf{x}|_{\mathcal{H}}$ and $\|\mathbf{x}\|_{\mathcal{H}}$ are equivalent and the inequality

$$\left\|\mathcal{T}(t)\right\|_{\mathcal{H}} \le \mathrm{const}\cdot\exp(-k\theta t), \quad t \ge 0$$

holds for some positive constant. □

Corollary 2.3. *Under the conditions of Theorem* 2.2 *for all* $\mathbf{x}_0 = (u_1, u_0)^\top \in \mathcal{D}(\mathbf{A})$ *vector-function*

$$\mathbf{x}(t) = \begin{pmatrix} w(t) \\ u(t) \end{pmatrix} = \mathcal{T}(t)\mathbf{x}_0 \in \mathcal{D}(\mathbf{A})$$

satisfies the first-order differential equation (2.2)*.* $u(t)$ *satisfies the second-order differential equation* (2.1) *with the initial conditions* (0.2) *and an inequality*

$$\|u(t)\|_1^2 + \|u'(t)\|^2 \le \mathrm{const}\cdot\exp\{-2k\theta t\}\left(\|u_0\|_1^2 + \|u_1\|^2\right)$$

holds for all $t \ge 0$*.*

Consider now a more strong assumption on the operator D:

(C) $D \in \mathcal{L}(H_1, H_{-1})$ and

$$\delta = \inf_{x \in H_1, x \ne 0} \frac{\operatorname{Re}(Dx, x)_{-1,1}}{\|x\|_1^2} > 0.$$

It is easy to show that the assumption (C) implies (B) and $\beta > a_0\delta$.

By $\|\tilde{S}\|$ and $\|D_2\|$ denote norms of the bounded operators $\tilde{S} \in \mathcal{L}(H_1, H_{-1})$ and $D_2 \in \mathcal{L}(H_1, H_{-1})$. Then for all $x \in H_1$

$$\|\tilde{S}x\|_{-1} \le \|\tilde{S}\| \cdot \|x\|_1, \quad \|D_2x\|_{-1} \le \|D_2\| \cdot \|x\|_1$$

[3]Obviously, the operator $(-\mathbf{A})$ is m-accretive as well.

Theorem 2.4. *Let the assumptions* (A) *and* (C) *are fulfilled and for some* $k \in (0, \beta)$ *and some* $p, q > 0$ *with* $p + q \le 1$

$$\omega_1' = a_0 \left(\frac{\delta}{k} - \frac{1}{4pk^2} \|\tilde{S}\|^2 - \frac{1}{4q} \|D_2\|^2 \right) \ge 1$$

Then the operator $\mathbf{A}$ *is a generator of a* C_0*-semigroup* $\mathcal{T}(t) = \exp\{t\mathbf{A}\}$ $(t \ge 0)$ *and*

$$\|\mathcal{T}(t)\|_{\mathcal{H}} \le \text{const} \cdot \exp(-tk\theta')$$

where

$$\theta' = \min \left\{ \frac{\omega_1' - 1}{2}, \frac{1 - p - q}{\omega_2} \right\} \ge 0$$

and ω_2 *is defined by* (2.4).

Proof. Consider on Hilbert space $\mathcal{H} = H \times H_1$ the scalar product $[\mathbf{x}, \mathbf{y}]_{\mathcal{H}}$. Then

$$-\frac{1}{k} \operatorname{Re}[\mathbf{A}\mathbf{x}, \mathbf{x}]_{\mathcal{H}} = \frac{1}{k}(D_1 x_1, x_1)_{-1,1} - \|x_1\|^2 + \|x_2\|_1^2 + \frac{1}{k} \operatorname{Im}(\tilde{S}x_1, x_2)_{-1,1} - \operatorname{Im}(D_2 x_1, x_2)_{-1,1}$$

(see the proof of Theorem 2.2). Since

$$\begin{aligned} |\operatorname{Im}(D_2 x_1, x_2)_{-1,1}| &\le |(D_2 x_1, x_2)_{-1,1}| \le \|D_2 x_1\|_{-1} \cdot \|x_2\|_1 \\ &\le \frac{1}{4q} \|D_2 x_1\|_{-1}^2 + q\|x_2\|_1^2 \le \frac{1}{4q} \|D_2\|^2 \cdot \|x_1\|_1^2 + q\|x_2\|_1^2 \end{aligned}$$

$$\begin{aligned} \frac{1}{k} |\operatorname{Im}(\tilde{S}x_1, x_2)_{-1,1}| &\le \left| \left(\frac{1}{k}\tilde{S}x_1, x_2 \right)_{-1,1} \right| \le \left\| \frac{1}{k}\tilde{S}x_1 \right\|_{-1} \cdot \|x_2\|_1 \\ &\le \frac{1}{4p} \left\| \frac{1}{k}\tilde{S}x_1 \right\|_{-1}^2 + p\|x_2\|_1^2 \le \frac{1}{4pk^2} \|\tilde{S}\|^2 \cdot \|x_1\|_1^2 + p\|x_2\|_1^2 \end{aligned}$$

and taking into account $(D_1 x, x)_{-1,1} \ge \delta \|x\|_1^2$ and $\|x\|_1^2 \ge a_0 \|x\|^2$ we obtain

$$\begin{aligned} &-\frac{1}{k} \operatorname{Re}[\mathbf{A}\mathbf{x}, \mathbf{x}]_{\mathcal{H}} \\ &\ge \frac{1}{k}(D_1 x_1, x_1)_{-1,1} - \|x_1\|^2 - \frac{\|\tilde{S}\|^2}{4pk^2} \cdot \|x_1\|_1^2 - \frac{\|D_2\|^2}{4q} \cdot \|x_1\|_1^2 + (1 - p - q)\|x_2\|_1^2 \\ &\ge \left(\frac{\delta}{k} - \frac{\|\tilde{S}\|^2}{4pk^2} - \frac{\|D_2\|^2}{4q} \right) \|x_1\|_1^2 - \|x_1\|^2 + (1 - p - q)\|x_2\|_1^2 \\ &\ge (\omega_1' - 1)\|x_1\|^2 + (1 - p - q)\|x_2\|_1^2. \end{aligned}$$

Using (2.5) we finally have

$$-\frac{1}{k} \operatorname{Re}[\mathbf{A}\mathbf{x}, \mathbf{x}]_{\mathcal{H}} \ge \theta' [\mathbf{x}, \mathbf{x}]_{\mathcal{H}}.$$

Thus an operator $(-\mathbf{A} - k\theta' I)$ in m-accretive (since $0 \in \rho(\mathbf{A})$) and

$$\rho(-\mathbf{A}) \supset \{\lambda \in \mathbb{C}, \ \operatorname{Re} \lambda < k\theta'\}.$$

Therefore, the operator $\mathbf{A}$ is a generator of a C_0-semigroup [4, 5] $\mathcal{T}(t) = \exp\{t\mathbf{A}\}$ $(t \geq 0)$ and

$$\left|\mathcal{T}(t)\right|_{\mathcal{H}} \leq \exp(-k\theta' t), \quad t \geq 0.$$

Since the norms $|\mathbf{x}|_{\mathcal{H}}$ and $\|\mathbf{x}\|_{\mathcal{H}}$ are equivalent then we have an inequality

$$\left\|\mathcal{T}(t)\right\|_{\mathcal{H}} \leq \mathrm{const} \cdot \exp(-k\theta' t), \quad t \geq 0$$

for some positive constant. □

Corollary 2.5. *Under the conditions of Theorem* 2.4 *for all* $\mathbf{x}_0 = (u_1, u_0)^\top \in \mathcal{D}(\mathbf{A})$ *a vector-valued function*

$$\mathbf{x}(t) = \begin{pmatrix} w(t) \\ u(t) \end{pmatrix} = \mathcal{T}(t)\mathbf{x}_0 \in \mathcal{D}(\mathbf{A})$$

satisfies the first-order differential equation (2.2). $u(t)$ *satisfies the second-order differential equation* (2.1) *with an initial conditions* (0.2) *and the inequality*

$$\|u(t)\|_1^2 + \|u'(t)\|^2 \leq \mathrm{const} \cdot \exp\{-2k\theta' t\}\left(\|u_0\|_1^2 + \|u_1\|^2\right)$$

holds for all $t \geq 0$.

3. Related spectral problem

Let us consider a quadric pencil associated with the differential equation (0.1)

$$L(\lambda) = \lambda^2 I + \lambda D + A \quad \lambda \in \mathbb{C}.$$

Since $D : H_1 \to H_{-1}$ it is more naturally to consider an extension of pencil

$$\tilde{L}(\lambda) = \lambda^2 I + \lambda D + \tilde{A}$$

mapping H_1 to H_{-1}. Moreover, $\tilde{L}(\lambda) \in \mathcal{L}(H_1, H_{-1})$ for all $\lambda \in \mathbb{C}$.

Definition 3.1. The resolvent set of the pencil $\tilde{L}(\lambda)$ is defined as

$$\rho(\tilde{L}) = \{\lambda \in \mathbb{C} \;:\; \exists \tilde{L}^{-1}(\lambda) \in \mathcal{L}(H_{-1}, H_1)\}$$

The spectrum of the pencil is $\sigma(\tilde{L}) = \mathbb{C} \backslash \rho(\tilde{L})$.

In [7, 16] it was proved that $\sigma(\tilde{L}) = \sigma(\mathbf{A})$ and for $\lambda \neq 0$

$$(\mathbf{A} - \lambda I)^{-1} = \begin{pmatrix} \lambda^{-1}\left(\tilde{L}^{-1}(\lambda)\tilde{A} - I\right) & -\tilde{L}^{-1}(\lambda) \\ \tilde{L}^{-1}(\lambda)\tilde{A} & -\lambda \tilde{L}^{-1}(\lambda) \end{pmatrix}$$

This result allows to obtain a localization of the pencil's spectrum in a half-plane.

Proposition 3.2.

1. *Under the conditions of Theorem* 2.2 *the spectrum of the pencil* $\tilde{L}(\lambda)$ *belongs to a half-plane*

$$\sigma(\tilde{L}) \subseteq \{\mathrm{Re} \leq -k\theta\}.$$

2. *Under the conditions of Theorem* 2.4 *the spectrum of the pencil* $\tilde{L}(\lambda)$ *belongs to a half-plane*

$$\sigma(\tilde{L}) \subseteq \{\mathrm{Re} \leq -k\theta'\}.$$

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Infinite Norm Decompositions of C*-algebras

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Abstract. In the given article the notion of infinite norm decomposition of a C*-algebra is investigated. The infinite norm decomposition is some generalization of Peirce decomposition. It is proved that the infinite norm decomposition of any C*-algebra is a C*-algebra. C*-factors with an infinite and a nonzero finite projection and simple purely infinite C*-algebras are constructed.

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Introduction

In the given article the notion of infinite norm decomposition of a C*-algebra is investigated. It is known that for any projection p of a unital C*-algebra A the next equality is valid $A = pAp \oplus pA(1-p) \oplus (1-p)Ap \oplus (1-p)A(1-p)$, where $\oplus$ is a direct sum of spaces. The infinite norm decomposition is some generalization of Peirce decomposition. First such infinite decompositions were introduced in [1] by the author.

In this article a unital C*-algebra A with an infinite orthogonal set $\{p_i\}$ of equivalent projections such that $\sup_i p_i = 1$, and the set $\sum_{ij}^{o} p_i A p_j = \{\{a_{ij}\} :$ for any indexes i, j, $a_{ij} \in p_i A p_j$, and $\|\sum_{k=1,\ldots,i-1}(a_{ki} + a_{ik}) + a_{ii}\| \to 0$ at $i \to \infty\}$ are considered. Note that all infinite sets like $\{p_i\}$ are supposed to be countable.

The main results of the given article are the next:

- For any C*-algebra A with an infinite orthogonal set $\{p_i\}$ of equivalent projections such that $\sup_i p_i = 1$ the set $\sum_{ij}^{o} p_i A p_j$ is a C*-algebra with the componentwise algebraic operations, the associative multiplication and the norm.
- There exist a C*-algebra A and different countable orthogonal sets $\{e_i\}$ and $\{f_i\}$ of equivalent projections in A such that $\sup_i e_i = 1$, $\sup_i f_i = 1$, $\sum_{ij}^{o} e_i A e_j \neq \sum_{ij}^{o} f_i A f_j$.

– If A is a W*-factor of type II_∞, then there exists a countable orthogonal set $\{p_i\}$ of equivalent projections in A such that $\sum_{ij}^o p_iAp_j$ is a C*-factor with a nonzero finite and an infinite projection. In this case $\sum_{ij}^o p_iAp_j$ is not a von Neumann algebra.
– If A is a W*-factor of type III, then for any countable orthogonal set $\{p_i\}$ of equivalent projections in A. The C*-subalgebra $\sum_{ij}^o p_iAp_j$ is simple and purely infinite. In this case $\sum_{ij}^o p_iAp_j$ is not a von Neumann algebra.
– There exists a C*-algebra A with an orthogonal set $\{p_i\}$ of equivalent projections such that $\sum_{ij}^o p_iAp_j$ is not a two-sided ideal of A.

1. Infinite norm decompositions

Lemma 1. *Let A be a C*-algebra, $\{p_i\}$ be an infinite orthogonal set of projections with the least upper bound 1 in the algebra A and let $\mathcal{A} = \{\{p_iap_j\} : a \in A\}$. Then,*

1) *the set $\mathcal{A}$ is a vector space with the next componentwise algebraic operations*

$$\lambda\{p_iap_j\} = \{p_i\lambda ap_j\}, \lambda \in \mathbb{C}$$
$$\{p_iap_j\} + \{p_ibp_j\} = \{p_i(a+b)p_j\}, a, b \in A,$$

2) *the algebra A and the vector space $\mathcal{A}$ can be identified in the sense of the next map*

$$\mathcal{I} : a \in A \to \{p_iap_j\} \in \mathcal{A}.$$

Proof. Item 1) of the lemma can be easily proved.

Proof of item 2): We assert that $\mathcal{I}$ is a one-to-one map. Indeed, it is clear, that for any $a \in A$ there exists a unique set $\{p_iap_j\}$, defined by the element a.

Suppose that there exist different elements a and b in A such that $p_iap_j = p_ibp_j$ for all i, j, i.e., $\mathcal{I}(a) = \mathcal{I}(b)$. Then $p_i(a-b)p_j = 0$ for all i and j. Observe that $p_i((a-b)p_j(a-b)^*) = ((a-b)p_j(a-b)^*)p_i = 0$ and $(a-b)p_j(a-b)^* \geq 0$ for all i, j. Therefore, the element $(a-b)p_j(a-b)^*$ commutes with every projection in $\{p_i\}$.

We prove $(a-b)p_j(a-b)^* = 0$. Indeed, there exists a maximal commutative $*$-subalgebra A_o of the algebra A, containing the set $\{p_i\}$ and the element $(a-b)p_j(a-b)^*$. Since $(a-b)p_j(a-b)^*p_i = p_i(a-b)p_j(a-b)^* = 0$ for any i, then the condition $(a-b)p_j(a-b)^* \neq 0$ contradicts the equality $\sup_i p_i = 1$.

Indeed, in this case $p_i \leq 1 - 1/\|(a-b)p_j(a-b)^*\|(a-b)p_j(a-b)^*$ for any i. Since by $(a-b)p_j(a-b)^* \neq 0$ we have $1 > 1-1/\|(a-b)p_j(a-b)^*\|(a-b)p_j(a-b)^*$, then we get a contradiction with $\sup_i p_i = 1$. Therefore $(a-b)p_j(a-b)^* = 0$.

Hence, since A is a C*-algebra, than $\|(a-b)p_j(a-b)^*\| = \|((a-b)p_j)((a-b)p_j)^*\| = \|((a-b)p_j)\|\|((a-b)p_j)^*\| = \|(a-b)p_j\|^2 = 0$ for any j. Therefore $(a-b)p_j = 0$, $p_j(a-b)^* = 0$ for any j. Analogously, we can get $p_j(a-b) = 0$, $(a-b)^*p_j = 0$ for any j. Hence the elements $a-b$, $(a-b)^*$ commute with every projection in $\{p_i\}$. Then there exists a maximal commutative $*$-subalgebra A_o of the algebra A, containing the set $\{p_i\}$ and the element $(a-b)(a-b)^*$. Since

$p_i(a-b)(a-b)^* = (a-b)(a-b)^*p_i = 0$ for any i, then the condition $(a-b)(a-b)^* \neq 0$ contradicts the equality $\sup_i p_i = 1$.

Therefore, $(a-b)(a-b)^* = 0$, $a-b=0$, i.e., $a=b$. Thus the map $\mathcal{I}$ is one-to-one. □

Lemma 2. *Let A be a C*-algebra, $\{p_i\}$ be an infinite orthogonal set of projections with the least upper bound* 1 *in the algebra A and $a \in A$. Then, if $p_i a p_j = 0$ for all i, j, then $a = 0$.*

Proof. Let $p \in \{p_i\}$. Observe that $p_i a p_j a^* = p_i(ap_ja^*) = ap_ja^*p_i = (ap_ja^*)p_i = 0$ for all i, j and $ap_ja^* = ap_jp_ja^* = (ap_j)(p_ja^*) = (ap_j)(ap_j)^* \geq 0$. Therefore, the element ap_ja^* commutes with all projections of the set $\{p_i\}$.

We prove $ap_ja^* = 0$. Indeed, there exists a maximal commutative $*$-subalgebra A_o of the algebra A, containing the set $\{p_i\}$ and the element ap_ja^*. Since $p_i(ap_ja^*) = (ap_ja^*)p_i = 0$ for any i, then the condition $ap_ja^* \neq 0$ contradicts the equality $\sup_i p_i = 1$ (see the proof of Lemma 1). Hence $ap_ja^* = 0$.

Hence, since A is a C*-algebra, then

$$\|ap_ja^*\| = \|(ap_j)(ap_j)^*\| = \|(ap_j)\|\|(ap_j)^*\| = \|ap_j\|^2 = 0$$

for any j. Therefore $ap_j = 0$, $p_ja^* = 0$ for any j. Analogously we have $p_ja = 0$, $a^*p_j = 0$ for any j. Hence the elements a, a^* commute with all projections of the set $\{p_i\}$. Then there exists a maximal commutative $*$-subalgebra A_o of the algebra A, containing the set $\{p_i\}$ and the element aa^*. Since $p_iaa^* = aa^*p_j = 0$ for any i, then the condition $aa^* \neq 0$ contradicts the equality $\sup_i p_i = 1$ (see the proof of Lemma 1). Hence $aa^* = 0$ and $a = 0$. □

Lemma 3. *Let A be a C*-algebra on a Hilbert space H, $\{p_i\}$ be an infinite orthogonal set of projections in A with the least upper bound* 1 *in the algebra $B(H)$ and $a \in A$. Then $a \geq 0$ if and only if for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ the inequality $pap \geq 0$ holds, where $p = \sum_{k=1}^n p_k$.*

Proof. By positivity of the operator $T : a \to bab, a \in A$ for any $b \in A$, if $a \geq 0$, then for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ the inequality $pap \geq 0$ holds.

Conversely, let $a \in A$. Suppose that for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ the inequality $pap \geq 0$ holds, where $p = \sum_{k=1}^n p_k$.

Let $a = c + id$ for some nonzero self-adjoint elements c, d in A. Then $(p_i + p_j)(c+id)(p_i+p_j) = (p_i+p_j)c(p_i+p_j) + i(p_i+p_j)d(p_i+p_j) \geq 0$ for all i, j. In this case the elements $(p_i+p_j)c(p_i+p_j)$ and $(p_i+p_j)d(p_i+p_j)$ are self-adjoint. Then $(p_i+p_j)d(p_i+p_j) = 0$ and $p_idp_j = 0$ for all i, j. Hence by Lemma 2 we have $d = 0$. Therefore $a = c = c^* = a^*$, i.e., $a \in A_{sa}$. Hence, a is a nonzero self-adjoint element in A. Let $b_n^\alpha = \sum_{kl=1}^n p_k^\alpha a p_l^\alpha$ for all natural numbers n and finite subsets $\{p_k^\alpha\}_{k=1}^n \subset \{p_i\}$. Then the set (b_n^α) ultraweakly converges to the element a.

Indeed, we have $A \subseteq B(H)$. Let $\{q_\xi\}$ be a maximal orthogonal set of minimal projections of the algebra $B(H)$ such, that $p_i = \sup_\eta q_\eta$ for some subset $\{q_\eta\} \subset \{q_\xi\}$ for any i. For arbitrary projections q and p in $\{q_\xi\}$ there exists a number $\lambda \in \mathbb{C}$ such, that $qap = \lambda u$, where u is an isometry in $B(H)$, satisfying the conditions $q = uu^*$, $p = u^*u$. Let $q_{\xi\xi} = q_\xi$, $q_{\xi\eta}$ be such element that $q_\xi = q_{\xi\eta}q_{\xi\eta}^*$, $q_\eta = q_{\xi\eta}^*q_{\xi\eta}$ for all different ξ and η. Then, let $\{\lambda_{\xi\eta}\}$ be a set of numbers such that $q_\xi a q_\eta = \lambda_{\xi\eta}q_{\xi\eta}$ for all ξ, η. In this case, since $q_\xi aa^* q_\xi = q_\xi(\sum_\eta \lambda_{\xi\eta}\bar{\lambda}_{\xi\eta})q_\xi < \infty$ we have the quantity of nonzero numbers of the set $\{\lambda_{\xi\eta}\}_\eta$ (ξth string of the infinite-dimensional matrix $\{\lambda_{\xi\eta}\}_{\xi\eta}$) is not greater then the countable cardinal number and the sequence (λ_n^ξ) of all these nonzero numbers converges to zero. Let v_{q_ξ} be a vector of the Hilbert space H which generates the minimal projection q_ξ. Then the set $\{v_{q_\xi}\}$ forms a complete orthonormal system of the space H. Let v be an arbitrary vector of the space H and μ_ξ be a coefficient of Fourier of the vector v, corresponding to v_{q_ξ} in relative to the complete orthonormal system $\{v_{q_\xi}\}$. Then, since $\sum_\xi \mu_\xi\bar{\mu}_\xi < \infty$ we have the quantity of all nonzero elements of the set $\{\mu_\xi\}_\xi$ is not greater then the countable cardinal number and the sequence (μ_n) of all these nonzero numbers converges to zero.

Let ν_ξ be the ξth coefficient of Fourier (corresponding to v_{q_ξ}) of the vector $a(v) \in H$ in relative to the complete orthonormal system $\{v_{q_\xi}\}$. Then $\nu_\xi = \sum_\eta \lambda_{\xi\eta}\mu_\eta$ and the scalar product $< a(v), v >$ is equal to the sum $\sum_\xi \nu_\xi\mu_\xi$. Since the element $a(v)$ belongs to H we have quantity of all nonzero elements in the set $\{\nu_\xi\}_\xi$ is not greater then the countable cardinal number and the sequence (ν_n) of all these nonzero numbers converges to zero.

Let ε be an arbitrary positive number. Then, since quantity of nonzero numbers of the sets $\{\mu_\xi\}_\xi$ and $\{\nu_\xi\}_\xi$ is not greater then the countable cardinal number and $\sum_\xi \nu_\xi\bar{\nu}_\xi < \infty$, $\sum_\xi \mu_\xi\bar{\mu}_\xi < \infty$ we have there exists $\{f_k\}_{k=1}^l \subset \{p_i\}$ such that for the set of indexes $\Omega_1 = \{\xi : \exists p \in \{f_k\}_{k=1}^l, q_\xi \leq p\}$ the next equality holds

$$\left|\sum_\xi \nu_\xi\mu_\xi - \sum_{\xi\in\Omega_1} \nu_\xi\mu_\xi\right| < \varepsilon.$$

Then, since quantity of nonzero numbers of the sets $\{\mu_\xi\}_\xi$ and $\{\lambda_{\xi\eta}\}_\eta$ is not greater then the countable cardinal number, and $\sum_\eta \lambda_{\xi\eta}\bar{\lambda}_{\xi\eta} < \infty$, $\sum_\xi \mu_\xi\bar{\mu}_\xi < \infty$ we have there exists $\{e_k\}_{k=1}^m \subset \{p_i\}$ such that for the set of indexes $\Omega_2 = \{\xi : \exists p \in \{e_k\}_{k=1}^m, q_\xi \leq p\}$ the next equality holds

$$\left|\sum_\eta \lambda_{\xi\eta}\mu_\eta - \sum_{\eta\in\Omega_2} \lambda_{\xi\eta}\mu_\eta\right| < \varepsilon.$$

Hence for the finite set $\{p_k\}_{k=1}^n = \{f_k\}_{k=1}^l \cup \{e_k\}_{k=1}^m$ and the set $\Omega = \{\xi : \exists p \in \{p_k\}_{k=1}^n, q_\xi \leq p\}$ of indexes we have

$$\left|\sum_\xi \nu_\xi\mu_\xi - \sum_{\xi\in\Omega}\left(\sum_{\eta\in\Omega} \lambda_{\xi\eta}\mu_\eta\right)\mu_\xi\right| < \varepsilon.$$

At the same time, $\langle(\sum_{kl=1}^n p_k a p_l)(v), v\rangle = \sum_{\xi\in\Omega}(\sum_{\eta\in\Omega} \lambda_{\xi\eta}\mu_\eta)\mu_\xi$. Therefore,

$$\left|\langle a(v), v\rangle - \left\langle\left(\sum_{kl=1}^n p_k a p_l\right)(v), v\right\rangle\right| < \varepsilon.$$

Hence, since the vector v and the number ε are chosen arbitrarily we have the net (b_n^α) ultraweakly converges to the element a.

Now there exists a maximal orthogonal set $\{e_\xi\}$ of minimal projections of the algebra $B(H)$ of all bounded linear operators on H such that the element a and the set $\{e_\xi\}$ belong to some maximal commutative $*$-subalgebra A_o of the algebra $B(H)$. We have for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ and $e \in \{e_\xi\}$ the inequality $e(\sum_{kl=1}^n p_k a p_l)e \geq 0$ holds by the positivity of the operator $T : b \to ebe, b \in A$.

By the previous part of the proof the net $(e_\xi b_n^\alpha e_\xi)_{\alpha n}$ ultraweakly converges to the element $e_\xi a e_\xi$ for any index ξ. Then we have $e_\xi b_n^\alpha e_\xi \geq 0$ for all n and α. Therefore, the ultraweak limit $e_\xi a e_\xi$ of the net $(e_\xi b_n^\alpha e_\xi)_{\alpha n}$ is a nonnegative element. Hence $e_\xi a e_\xi \geq 0$. Therefore, since e_ξ is chosen arbitrarily we have $a \geq 0$. □

Lemma 4. *Let A be a C*-algebra on a Hilbert space H, $\{p_i\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$ and $a \in A$. Then*

$$\|a\| = \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}.$$

Proof. The inequality $-\|a\|1 \leq a \leq \|a\|1$ holds. Then $-\|a\|p \leq pap \leq \|a\|p$ for all natural numbers n and finite subsets $\{p_k^\alpha\}_{k=1}^n \subset \{p_i\}$, where $p = \sum_{k=1}^n p_k$. Therefore

$$\|a\| \geq \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}.$$

At the same time, since the finite subset $\{p_k\}_{k=1}^n$ of $\{p_i\}$ is chosen arbitrarily and by Lemma 6 we have

$$\|a\| = \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}.$$

Otherwise, if

$$\|a\| > \lambda = \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}$$

then by Lemma 3 $-\lambda 1 \leq a \leq \lambda 1$. But the last inequality is a contradiction. □

Lemma 5. *Let A be a C*-algebra on a Hilbert space H, $\{p_i\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$, and let $\mathcal{A} = \{\{p_i a p_j\} : a \in A\}$. Then,*

1) *the vector space $\mathcal{A}$ is a unit-order space with respect to the order $\{p_i a p_j\} \geq 0$ ($\{p_i a p_j\} \geq 0$ if for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ the inequality $pap \geq 0$ holds, where $p = \sum_{k=1}^n p_k$) and the norm*

$$\|\{p_i a p_j\}\| = \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}.$$

2) *the algebra A and the unit-order space $\mathcal{A}$ can be identified as unit-order spaces in the sense of the map*

$$\mathcal{I} : a \in A \to \{p_i a p_j\} \in \mathcal{A}.$$

Proof. This lemma follows by Lemmas 1, 3 and 4. □

Remark. Observe that by Lemma 4 the order and the norm in the unit-order space $\mathcal{A} = \{\{p_i a p_j\} : a \in A\}$ can be defined as follows to: $\{p_i a p_j\} \geq 0$ if $a \geq 0$; $\|\{p_i a p_j\}\| = \|a\|$. By Lemmas 3 and 4 they are equivalent to the order and the norm, defined in Lemma 5, correspondingly.

Let A be a C*-algebra, $\{p_i\}$ be a countable orthogonal set of equivalent projections in A such that $\sup_i p_i = 1$ and

$$\sum_{ij}^{o} p_i A p_j = \Bigg\{\{a_{ij}\} : \text{for any indexes}\, i,j,\ a_{ij} \in p_i A p_j, \text{ and } \Bigg\|\sum_{k=1,\ldots,i-1} (a_{ki} + a_{ik}) + a_{ii}\Bigg\| \to 0 \,\text{at}\; i \to \infty\Bigg\}.$$

If we introduce a componentwise algebraic operations in this set then $\sum_{ij}^{o} p_i A p_j$ becomes a vector space. Also, note that $\sum_{ij}^{o} p_i A p_j$ is a vector subspace of $\mathcal{A}$. Observe that $\sum_{ij}^{o} p_i A p_j$ is a normed subspace of the algebra $\mathcal{A}$ and $\| \sum_{i,j=1}^{n} a_{ij} - \sum_{i,j=1}^{n+1} a_{ij}\| \to 0$ at $n \to \infty$ for any $\{a_{ij}\} \in \sum_{ij}^{o} p_i A p_j$.

Let $\sum_{ij}^{o} a_{ij} := \lim_{n\to\infty} \sum_{i,j=1}^{n} a_{ij}$ for any $\{a_{ij}\} \in \sum_{ij}^{o} p_i A p_j$ and

$$C^*(\{p_i A p_j\}_{ij}) := \Bigg\{\sum_{ij}^{o} a_{ij} : \{a_{ij}\} \in \sum_{ij}^{o} p_i A p_j\Bigg\}.$$

Then $C^*(\{p_i A p_j\}_{ij}) \subseteq A$. By Lemma 5 A and $\mathcal{A}$ can be identified. We observe that, the normed spaces $\sum_{ij}^{o} p_i A p_j$ and $C^*(\{p_i A p_j\}_{ij})$ can also be identified. Further, without loss of generality we will use these identifications.

Theorem 6. *Let A be a unital C*-algebra, $\{p_i\}$ be a countable orthogonal set of equivalent projections in A and $\sup_i p_i = 1$. Then $\sum_{ij}^{o} p_i A p_j$ is a C*-subalgebra of A with the componentwise algebraic operations, the associative multiplication and the norm.*

Proof. We have $\sum_{ij}^{o} p_i A p_j$ is a normed subspace of the algebra A.

Let (a_n) be a sequence of elements in $\sum_{ij}^{o} p_i A p_j$ such that (a_n) norm converges to some element $a \in A$. We have $p_i a_n p_j \to p_i a p_j$ at $n \to \infty$ for all i and j. Hence $p_i a p_j \in p_i A p_j$ for all i, j. Let $b^n = \sum_{k=1}^{n} (p_{n-1} a p_k + p_k a p_{n-1}) + p_n a p_n$ and $c_m^n = \sum_{k=1}^{n} (p_{n-1} a_m p_k + p_k a_m p_{n-1}) + p_n a_m p_n$ for any n. Then $c_m^n \to b^n$ at $m \to \infty$. It should be proven that $\|b_n\| \to 0$ at $n \to \infty$.

Let $\varepsilon \in \mathbb{R}_+$. Then there exists m_o such that $\|a - a_m\| < \varepsilon$ for any $m > m_o$. Also for all n and $\{p_k\}_{k=1}^{n} \subset \{p_i\}$ $\|(\sum_{k=1}^{n} p_k)(a - a_m)(\sum_{k=1}^{n} p_k)\| < \varepsilon$. Hence $\|b^n - c_m^n\| < 2\varepsilon$ for any $m > m_o$. At the same time, $\|b^n - c_{m_1}^n\| < 2\varepsilon$, $\|b^n - c_{m_2}^n\| < 2\varepsilon$

for all $m_o < m_1$, m_2. Since $(a_n) \subset \sum_{ij}^o p_iAp_j$ then for any m $\|c_m^n\| \to 0$ at $n \to \infty$. Hence, since $\|c_{m_1}^n\| \to 0$ and $\|c_{m_2}^n\| \to 0$ at $n \to \infty$ we have there exists n_o such that $\|c_{m_1}^n\| < \varepsilon$, $\|c_{m_2}^n\| < \varepsilon$ and $\|c_{m_1}^n + c_{m_2}^n\| < 2\varepsilon$ for any $n > n_o$. Then $\|2b_n\| = \|b^n - c_{m_1}^n + c_{m_1}^n + c_{m_2}^n + b^n - c_{m_2}^n\| \leq \|b^n - c_{m_1}^n\| + \|c_{m_1}^n + c_{m_2}^n\| + \|b^n - c_{m_2}^n\| < 2\varepsilon + 2\varepsilon + 2\varepsilon = 6\varepsilon$ for any $n > n_o$, i.e., $\|b_n\| < 3\varepsilon$ for any $n > n_o$. Since ε is chosen arbitrarily we have $\|b_n\| \to 0$ at $n \to \infty$. Therefore $a \in \sum_{ij}^o p_iAp_j$. Since the sequence (a_n) is chosen arbitrarily we have $\sum_{ij}^o p_iAp_j$ is a Banach space.

Let $\{a_{ij}\}$, $\{b_{ij}\}$ be arbitrary elements of the Banach space $\sum_{ij}^o p_iAp_j$. Let $a_m = \sum_{kl=1}^m a_{kl}$, $b_m = \sum_{kl=1}^m b_{kl}$ for all natural numbers m. We have the sequence (a_m) converges to $\{a_{ij}\}$ and the sequence (b_m) converges to $\{b_{ij}\}$ in $\sum_{ij}^o p_iAp_j$. Also for all n and m $a_mb_n \in \sum_{ij}^o p_iAp_j$. Then for any n the sequence (a_mb_n) converges to $\{a_{ij}\}b_n$ at $m \to \infty$. Hence $\{a_{ij}\}b_n \in \sum_{ij}^o p_iAp_j$. Note that $\sum_{ij}^o p_iAp_j \subseteq A$. Therefore for any $\varepsilon \in \mathbb{R}_+$ there exists n_o such that $\|\{a_{ij}\}b_{n+1} - \{a_{ij}\}b_n\| \leq \|\{a_{ij}\}\|\|b_{n+1} - b_n\| \leq \varepsilon$ for any $n > n_o$. Hence the sequence $(\{a_{ij}\}b_n)$ converges to $\{a_{ij}\}\{b_{ij}\}$ at $n \to \infty$. Since $\sum_{ij}^o p_iAp_j$ is a Banach space then $\{a_{ij}\}\{b_{ij}\} \in \sum_{ij}^o p_iAp_j$. Since $\sum_{ij}^o p_iAp_j \subseteq A$ we have $\sum_{ij}^o p_iAp_j$ is a C*-algebra. □

Let H be an infinite-dimensional Hilbert space, $B(H)$ be the algebra of all bounded linear operators. Let $\{p_i\}$ be a countable orthogonal set of equivalent projections in $B(H)$ and $\sup_i p_i = 1$. Let $\{\{p_j^i\}_j\}_i$ be the set of infinite subsets of $\{p_i\}$ such that for all distinct ξ and η $\{p_j^\xi\}_j \cap \{p_j^\eta\}_j = \oslash$, $|\{p_j^\xi\}_j| = |\{p_j^\eta\}_j|$ and $\{p_i\} = \cup_i\{p_j^i\}_j$. Then let $q_i = \sup_j p_j^i$ in $B(H)$ for all i. Then $\sup_i q_i = 1$ and $\{q_i\}$ be a countable orthogonal set of equivalent projections. Then we say that the countable orthogonal set $\{q_i\}$ of equivalent projections *is defined by the set* $\{p_i\}$ in $B(H)$. We have the next corollary.

Corollary 7. *Let A be a unital C*-algebra on a Hilbert space H, $\{p_i\}$ be a countable orthogonal set of equivalent projections in A and* $\sup_i p_i = 1$. *Let $\{q_i\}$ be a countable orthogonal set of equivalent projections in $B(H)$ defined by the set $\{p_i\}$ in $B(H)$. Then $\sum_{ij}^o q_iAq_j$ is a C*-subalgebra of the algebra A.*

Proof. Let $\{\{p_j^i\}_j\}_i$ be the set of infinite subsets of $\{p_i\}$ such that for all distinct ξ and η $\{p_j^\xi\}_j \cap \{p_j^\eta\}_j = \oslash$, $|\{p_j^\xi\}_j| = |\{p_j^\eta\}_j|$ and $\{p_i\} = \cup_i\{p_j^i\}_j$. Then let $q_i = \sup_j p_j^i$ in $B(H)$ for all i. Then we have for all i and j $q_iAq_j = \{\{p_\xi^i a p_\eta^j\}_{\xi\eta} : a \in A\}$. Hence $q_iAq_j \subset A$ for all i and j.

The rest part of the proof is the repeating of the proof of Theorem 6. □

Example. 1. Let $\mathcal{M}$ be the closure on the norm of the inductive limit $\mathcal{M}_o$ of the inductive system

$$C \to M_2(C) \to M_3(C) \to M_4(C) \to \cdots,$$

where $M_n(C)$ is mapped into the upper left corner of $M_{n+1}(C)$. Then $\mathcal{M}$ is a C*-algebra ([1]). The algebra $\mathcal{M}$ contains the minimal projections of the form e_{ii}, where e_{ij} is an infinite-dimensional matrix, whose (i,i)th component is 1 and the rest components are zeros. These projections form the countable orthogonal set $\{e_{ii}\}_{i=1}^{\infty}$ of minimal projections. Let

$$M_n^o(\mathbb{C}) = \left\{ \sum_{ij} \lambda_{ij} e_{ij} : \lambda_{ij} \in \mathbb{C} \,\text{for any indexes}\, i, j \,\text{and}\right.$$
$$\left. \left\| \sum_{k=1,\ldots,i-1} (\lambda_{ki} e_{ki} + \lambda_{ik} e_{ik}) + \lambda_{ii} e_{ii} \right\| \to 0 \,\text{at}\, i \to \infty \right\}.$$

Then $\mathbb{C} \cdot 1 + M_n^o(\mathbb{C}) = \mathcal{M}$ (see [2]) and by Theorem 6 $M_n^o(\mathbb{C})$ is a simple C*-algebra. Note that there exists a mistake in the formulation of Theorem 3 in [2]. $\mathbb{C} \cdot 1 + M_n^o(\mathbb{C})$ is a C*-algebra. But the algebra $\mathbb{C} \cdot 1 + M_n^o(\mathbb{C})$ is not simple. Because $\mathbb{C} \cdot 1 + M_n^o(\mathbb{C}) \neq M_n^o(\mathbb{C})$ and $M_n^o(\mathbb{C})$ is an ideal of the algebra $\mathbb{C} \cdot 1 + M_n^o(\mathbb{C})$, i.e., $[\mathbb{C} \cdot 1 + M_n^o(\mathbb{C})] \cdot M_n^o(\mathbb{C}) \subseteq M_n^o(\mathbb{C})$.

2. There exist a C*-algebra A and different countable orthogonal sets $\{e_i\}$ and $\{f_i\}$ of equivalent projections in A such that $\sup_i e_i = 1$, $\sup_i f_i = 1$, $\sum_{ij}^o e_i A e_j \neq \sum_{ij}^o f_i A f_j$. Indeed, let H be an infinite-dimensional Hilbert space, $B(H)$ be the algebra of all bounded linear operators. Let $\{p_i\}$ be a countable orthogonal set of equivalent projections in $B(H)$ and $\sup_i p_i = 1$. Then $\sum_{ij}^o p_i B(H) p_j \subset B(H)$. Let $\{\{p_j^i\}_j\}_i$ be the set of infinite subsets of $\{p_i\}$ such that for all distinct ξ and η $\{p_j^\xi\}_j \cap \{p_j^\eta\}_j = \oslash$, $|\{p_j^\xi\}_j| = |\{p_j^\eta\}_j|$ and $\{p_i\} = \cup_i \{p_j^i\}_j$. Then let $q_i = \sup_j p_j^i$ for all i. Then $\sup_i q_i = 1$ and $\{q_i\}$ be a countable orthogonal set of equivalent projections. We assert that $\sum_{ij}^o p_i B(H) p_j \neq \sum_{ij}^o q_i B(H) q_j$. Indeed, let $\{x_{ij}\}$ be a set of matrix units constructed by the infinite set $\{p_j^1\}_j \in \{\{p_j^i\}_j\}_i$, i.e., for all i, j, $x_{ij} x_{ij}^* = p_i^1$, $x_{ij}^* x_{ij} = p_j^1$, $x_{ii} = p_i^1$. Then the von Neumann algebra $\mathcal{N}$ generated by the set $\{x_{ij}\}$ is isometrically isomorphic to $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We note that $\mathcal{N}$ is not a subset of $\sum_{ij}^o p_i B(H) p_j$. At the same time, $\mathcal{N} \subseteq \sum_{ij}^o q_i B(H) q_j$ and $\sum_{ij}^o p_i^1 \mathcal{N} p_j^1 \subseteq \sum_{ij}^o p_i B(H) p_j$.

Theorem 8. *Let A be a unital simple C*-algebra on a Hilbert space H, $\{p_i\}$ be a countable orthogonal set of equivalent projections in A and $\sup_i p_i = 1$. Let $\{q_i\}$ be a countable orthogonal set of equivalent projections in $B(H)$ defined by the set $\{p_i\}$ in $B(H)$. Then $\sum_{ij}^o q_i A q_j$ is a simple C*-algebra.*

Proof. By Theorem 6 $\sum_{ij}^o p_i A p_j$ is a C*-algebra. Let $\{\{p_j^i\}_j\}_i$ be the set of infinite subsets of $\{p_i\}$ such that for all distinct ξ and η $\{p_j^\xi\}_j \cap \{p_j^\eta\}_j = \oslash$, $|\{p_j^\xi\}_j| = |\{p_j^\eta\}_j|$ and $\{p_i\} = \cup_i \{p_j^i\}_j$. Then let $q_i = \sup_j p_j^i$ in $B(H)$, for all i. Then we have $q_i A q_j = \{\{p_\xi^i a p_\xi^j\} : a \in A\}$ for all i and j. Hence $q_i A q_j \subset A$ for all i and j. By Corollary 7 $\sum_{ij}^o q_i A q_j$ is a C*-algebra.

Since projections of the set $\{p_i\}$ are pairwise equivalent we have the projection q_i is equivalent to $1 \in A$ for any i. Hence $q_i A q_i \cong A$ and $q_i A q_i$ is a simple C*-algebra for any i.

Let q be an arbitrary projection in $\{q_i\}$. Then qAq is a C*-subalgebra of $\sum_{ij}^{o} q_iAq_j$. Let I be a closed two-sided ideal of the algebra $\sum_{ij}^{o} q_iAq_j$. Then $IqAq \subset I$ and $Iq \cdot qAq \subset Iq$. Therefore $qIqqAq \subseteq qIq$, that is qIq is a closed two-sided ideal of the subalgebra qAq. Since qAq is simple then $qIq = qAq$.

Let q_1, q_2 be arbitrary projections in $\{q_i\}$. We assert that $q_1Iq_2 = q_1Aq_2$ and $q_2Iq_1 = q_2Aq_1$. Indeed, we have the projection $q_1 + q_2$ is equivalent to $1 \in A$. Let $e = q_1 + q_2$. Then $eAe \cong A$ and eAe is a simple C*-algebra. At the same time we have eAe is a subalgebra of $\sum_{ij}^{o} q_iAq_j$ and I is a two-sided ideal of $\sum_{ij}^{o} q_iAq_j$. Hence $IeAe \subset I$ and $Ie \cdot eAe \subset Ie$. Therefore $eIeeAe \subseteq eIe$, that is eIe is a closed two-sided ideal of the subalgebra eAe. Since eAe is simple then $eIe = eAe$. Hence $q_1Iq_2 = q_1Aq_2$ and $q_2Iq_1 = q_2Aq_1$. Therefore $q_iIq_j = q_iAq_j$ for all i and j. We have I is norm closed. Hence $I = \sum_{ij}^{o} q_iAq_j$, i.e., $\sum_{ij}^{o} q_iAq_j$ is a simple C*-algebra. □

2. Applications

Definition. A C*-algebra is called a C*-factor, if it does not have nonzero proper two-sided ideals I and J such that $I \cdot J = \{0\}$, where $I \cdot J = \{ab : a \in I, b \in J\}$.

Theorem 9. *Let $\mathcal{N}$ be a W*-factor of type II_∞ on a Hilbert space H, $\{p_i\}$ be a countable orthogonal set of equivalent projections in $\mathcal{N}$ and* $\sup_i p_i = 1$. *Then for any countable orthogonal set $\{q_i\}$ of equivalent projections in $B(H)$ defined by the set $\{p_i\}$ in $B(H)$ the C*-algebra $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is a C*-factor with a nonzero finite and an infinite projection. In this case $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is not a von Neumann algebra.*

Proof. By the definition of the set $\{q_i\}$ we have $\sup_i q_i = 1$ and $\{q_i\}$ be a countable orthogonal set of equivalent *infinite* projections. By Theorem 6 we have $\sum_{ij}^{o} q_i\mathcal{N}p_j$ is a C*-subalgebra of $\mathcal{N}$. Let q be a nonzero finite projection of $\mathcal{N}$. Then there exists a projection $p \in \{q_i\}$ such that $qp \neq 0$. We have $q\mathcal{N}q$ is a finite von Neumann algebra. Let $x = pq$. Then $x\mathcal{N}x^*$ is a weakly closed C*-subalgebra. Note that the algebra $x\mathcal{N}x^*$ has a center-valued faithful trace. Let e be a nonzero projection of the algebra $x\mathcal{N}x^*$. Then $ep = e$ and $e \in p\mathcal{N}p$. Hence $e \in \sum_{ij}^{o} q_i\mathcal{N}q_j$. We have the weak closure of $\sum_{ij}^{o} q_i\mathcal{N}q_j$ in the algebra $\mathcal{N}$ coincides with this algebra $\mathcal{N}$. Then by the weak continuity of the multiplication $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is a C*-factor. Note since $1 \notin \sum_{ij}^{o} q_i\mathcal{N}q_j$ then $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is not weakly closed in $\mathcal{N}$. Hence the C*-factor $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is not a von Neumann algebra. □

Remark. Note that, in the article [3] a simple C*-algebra with an infinite and a nonzero finite projection have been constructed by M.Rørdam. In the next corollary we construct a simple purely infinite C*-algebra. Note that simple purely infinite C*-algebras are considered and investigated, in particular, in [4] and [5].

Theorem 10. *Let $\mathcal{N}$ be a W*-factor of type III on a Hilbert space H. Then for any countable orthogonal set $\{p_i\}$ of equivalent projections in $\mathcal{N}$ such that* $\sup_i p_i = 1$, *$\sum_{ij}^{o} p_i\mathcal{N}p_j$ is a simple purely infinite C*-algebra. In this case $\sum_{ij}^{o} p_i\mathcal{N}p_j$ is not a von Neumann algebra.*

Proof. Let p_{i_o} be a projection in $\{p_i\}$. We have the projection p_{i_o} can be exhibited as a least upper bound of a countable orthogonal set $\{p^j_{i_o}\}_j$ of equivalent projections in $\mathcal{N}$. Then for any i the projection p_i has a countable orthogonal set $\{p^j_i\}_j$ of equivalent projections in $\mathcal{N}$ such that the set $\bigcup_i \{p^j_i\}_j$ is a countable orthogonal set of equivalent projections in $\mathcal{N}$. Hence the set $\{p_i\}$ is defined by the set $\bigcup_i \{p^j_i\}_j$ in $B(H)$ (in $\mathcal{N}$). Hence by Theorem 8 $\sum^o_{ij} p_i\mathcal{N}p_j$ is a simple C*-algebra. Note, since $1 \notin \sum^o_{ij} p_i\mathcal{N}p_j$ we have $\sum^o_{ij} p_i\mathcal{N}p_j$ is not weakly closed in $\mathcal{N}$. Hence $\sum^o_{ij} p_i\mathcal{N}p_j$ is not a von Neumann algebra.

Suppose there exists a nonzero finite projection q in $\sum^o_{ij} p_i\mathcal{N}p_j$. Then there exists a projection $p \in \{p_i\}$ such that $qp \neq 0$. We have $q(\sum^o_{ij} p_i\mathcal{N}p_j)q$ is a finite C*-algebra. Let $x = pq$. Then $x\mathcal{N}x^*$ is a C*-subalgebra. Moreover $x\mathcal{N}x^*$ is weakly closed and $x\mathcal{N}x^* \subset p\mathcal{N}p$. Hence $x\mathcal{N}x^*$ has a center-valued faithful trace. Then $x\mathcal{N}x^*$ is a finite von Neumann algebra with a center-valued faithful normal trace. Let e be a nonzero projection of the algebra $x\mathcal{N}x^*$. Then $ep = e$ and $e \in p\mathcal{N}p$. Hence $e \in \mathcal{N}$. This is a contradiction. □

Example. Let H be a separable Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Let $\{q_i\}$ be a maximal orthogonal set of equivalent minimal projections in $B(H)$. Then $\sum^o_{ij} q_iB(H)q_j$ is a two-sided closed ideal of the algebra $B(H)$. Using the set $\{q_i\}$ we construct a countable orthogonal set $\{p_i\}$ of equivalent infinite projections such that $\sup_i p_i = 1$. Let $\{\{q^i_j\}_j\}_i$ be the countable set of countable subsets of $\{q_i\}$ such that for all distinct i_1 and i_2 $\{q^{i_1}_j\}_j \cap \{p^{i_2}_j\}_j = \oslash$ and $\{q_i\} = \cup_i\{q^i_j\}_j$. Then let $p_i = \sup_j q^i_j$ for all i. Then $\sup_i p_i = 1$ and $\{p_i\}$ is a countable orthogonal set of equivalent infinite projections in $B(H)$ defined by $\{q_i\}$ in $B(H)$.

Let $\{q^{ij}_{nm}\}$ be the set of matrix units constructed by the set $\{\{q^i_j\}_j\}_i$, i.e., $q^{ij}_{nm}{q^{ij}_{nm}}^* = q^i_n$, ${q^{ij}_{nm}}^* q^{ij}_{nm} = q^j_m$, $q^{ii}_{nn} = q^i_n$ for all i, j,n,m. Then let $a = \{a^{ij}_{nm}q^{ij}_{nm}\}$ be the decomposition of the element $a \in B(H)$, where the components a^{ij}_{nm} are defined as follows

$$a^{11}_{11} = \lambda, a^{21}_{12} = \lambda, a^{31}_{13} = \lambda, \ldots, a^{n1}_{1n} = \lambda, \ldots,$$

and the rest components a^{ij}_{nm} are zero, i.e., $a^{ij}_{nm} = 0$. Then $p_1a = a$. Then, since $a \notin \sum^o_{ij} p_iB(H)p_j$ and $p_1 \in \sum^o_{ij} p_iB(H)p_j$ we have $\sum^o_{ij} p_iB(H)p_j$ is not a two-sided ideal of $B(H)$. But by theorem 6 $\sum^o_{ij} p_iB(H)p_j$ is a C*-algebra. Hence there exists a C*-algebra A with an orthogonal set $\{p_i\}$ of equivalent projections such that $\sum^o_{ij} p_iAp_j$ is not a two-sided ideal of A.

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Canonical Transfer-function Realization for Schur-Agler-class Functions on Domains with Matrix Polynomial Defining Function in $\mathbb{C}^n$

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Abstract. It is well known that a Schur-class function $S(z)$, i.e., a holomorphic function on the unit disk whose values are contraction operators between two Hilbert spaces $\mathcal{U}$ (the input space) and $\mathcal{Y}$ (the output space), can be written as the characteristic function $S(z) = D + zC(I - zA)^{-1}B$ of the unitary colligation $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ (or as the transfer function of the associated conservative linear system) where $\mathbf{U}$ defines a unitary operator from $\mathcal{X} \oplus \mathcal{U}$ to $\mathcal{X} \oplus \mathcal{Y}$ where the Hilbert space $\mathcal{X}$ is an appropriately chosen state space. Moreover, this transfer function is essentially uniquely determined if $\mathbf{U}$ is also required to satisfy a certain minimality condition ($\mathbf{U}$ should be "closely-connected"). In addition, by choosing the state space $\mathcal{X}$ to be the two-component de Branges-Rovnyak reproducing kernel Hilbert space $\mathcal{H}(\widehat{K})$, one can arrive at a unique canonical functional-model form for a $\mathbf{U}$ meeting the minimality requirement. Recent work of the authors and others has extended the notion of Schur class and transfer-function representation for Schur-class functions to several-variable complex domains with matrix-polynomial defining function. In this setting the term "Schur-Agler class" is used since one also imposes that a certain von Neumann inequality be satisfied. In this article we develop an analogue of the two-component de Branges-Rovnyak reproducing kernel Hilbert space for this more general setting and thereby arrive at a two-component canonical functional model colligation for the analogue of closely-connected unitary transfer-function realization for this Schur-Agler class. A number of new technical issues appear in this setting which are not present in the classical case.

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1. Introduction

Let $\mathcal{U}$ and $\mathcal{Y}$ be two Hilbert spaces and let $\mathcal{L}(\mathcal{U},\mathcal{Y})$ be the space of all bounded linear operators between $\mathcal{U}$ and $\mathcal{Y}$. We also let $H^2_{\mathcal{U}}$ be the standard Hardy space of the $\mathcal{U}$-valued holomorphic functions on the unit disk $\mathbb{D}$. The operator-valued version of the classical Schur class $\mathcal{S}(\mathcal{U},\mathcal{Y})$ is defined to be the set of all holomorphic, contractive $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued functions on $\mathbb{D}$. With any function $S\in\mathcal{S}(\mathcal{U},\mathcal{Y})$, one can associate the following three operator-valued kernels

$$K_S(z,\zeta)=\frac{I_{\mathcal{Y}}-S(z)S(\zeta)^*}{1-z\overline{\zeta}},\quad \widetilde{K}_S(z,\zeta)=\frac{I_{\mathcal{U}}-S(z)^*S(\zeta)}{1-\overline{z}\zeta},\tag{1.1}$$

$$\widehat{K}_S(z,\zeta)=\begin{bmatrix} K_S(z,\zeta) & \dfrac{S(z)-S(\zeta)}{z-\zeta}\\ \dfrac{S(z)^*-S(\zeta)^*}{\overline{z}-\overline{\zeta}} & \widetilde{K}_S(z,\zeta)\end{bmatrix}.\tag{1.2}$$

The following equivalent characterizations of the Schur class are well known.

Theorem 1.1. *Let $S\colon\mathbb{D}\to\mathcal{L}(\mathcal{U},\mathcal{Y})$ be given. Then the following are equivalent:*

(1a) *$S\in\mathcal{S}(\mathcal{U},\mathcal{Y})$, i.e., S is holomorphic on $\mathbb{D}$ with $\|S(z)\|\le 1$ for all $z\in\mathbb{D}$.*

(1b) *S satisfies the von Neumann inequality: $\|S(T)\|\le 1$ for any strictly contractive operator T on a Hilbert space $\mathcal{H}$, where $S(T)$ is defined by*

$$S(T)=\sum_{n=0}^{\infty}S_n\otimes T^n\in\mathcal{L}(\mathcal{U}\otimes\mathcal{H},\mathcal{Y}\otimes\mathcal{H})\quad \textit{if}\quad S(z)=\sum_{n=0}^{\infty}S_nz^n.$$

(2a) *The kernel $K_S(z,\zeta)$ is positive on $\mathbb{D}\times\mathbb{D}$.*

(2b) *The kernel $\widetilde{K}_S(z,\zeta)$ is positive on $\mathbb{D}\times\mathbb{D}$.*

(2c) *The kernel $\widehat{K}(z,\zeta)$ is positive on $\mathbb{D}\times\mathbb{D}$.*

(3) *There is an auxiliary Hilbert space $\mathcal{X}$ and a unitary connecting operator (or colligation) $\mathbf{U}=\left[\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right]\colon\left[\begin{smallmatrix}\mathcal{X}\\\mathcal{U}\end{smallmatrix}\right]\to\left[\begin{smallmatrix}\mathcal{X}\\\mathcal{Y}\end{smallmatrix}\right]$ so that $S(z)$ can be expressed as*

$$S(z)=D+zC(I-zA)^{-1}B.\tag{1.3}$$

(4) *$S(z)$ has a realization as in* (1.3) *where the connecting operator $\mathbf{U}$ is any one of* (i) *isometric,* (ii) *coisometric, or* (iii) *contractive.*

The function on the right-hand side of (1.3) is called the *transfer function* of the colligation $\mathbf{U}=\left[\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right]$ and thus, statement (3) in Theorem 1.1 asserts that every Schur-class function can be realized as the transfer function of a unitary colligation.

Two colligations $\mathbf{U}=\left[\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right]\colon\left[\begin{smallmatrix}\mathcal{X}\\\mathcal{U}\end{smallmatrix}\right]\to\left[\begin{smallmatrix}\mathcal{X}\\\mathcal{Y}\end{smallmatrix}\right]$ and $\mathbf{U}'=\left[\begin{smallmatrix}A'&B'\\C'&D'\end{smallmatrix}\right]\colon\left[\begin{smallmatrix}\mathcal{X}'\\\mathcal{U}\end{smallmatrix}\right]\to\left[\begin{smallmatrix}\mathcal{X}'\\\mathcal{Y}\end{smallmatrix}\right]$ are called *unitarily equivalent* if there is a unitary operator $U\colon\mathcal{X}\to\mathcal{X}'$ so that

$$\begin{bmatrix}U&0\\0&I_{\mathcal{Y}}\end{bmatrix}\begin{bmatrix}A&B\\C&D\end{bmatrix}=\begin{bmatrix}A'&B'\\C'&D'\end{bmatrix}\begin{bmatrix}U&0\\0&I_{\mathcal{U}}\end{bmatrix}.$$

It is readily seen that if two colligations are unitarily equivalent, then their transfer functions coincide. The converse is true under certain minimality conditions which we now recall. In what follows, the symbol $\bigvee$ stands for the closed linear span.

Definition 1.2. The colligation $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is called

1) *observable* (or closely outer-connected) if the pair (C, A) is observable, i.e., if $\bigvee_{n\geq 0} \operatorname{Ran} A^{*n}C^* = \mathcal{X}$;
2) *controllable* (or closely inner-connected) if the pair (B, A) is controllable, i.e., if $\bigvee_{n\geq 0} \operatorname{Ran} A^n B = \mathcal{X}$;
3) *closely connected* if $\bigvee_{n\geq 0} \{\operatorname{Ran} A^n B, \operatorname{Ran} A^{*n}C^*\} = \mathcal{X}$.

The positive kernel functions K_S, $\widetilde{K}_S$ and $\widehat{K}_S$ given by (1.1), (1.2) and the reproducing kernel Hilbert spaces $\mathcal{H}(K_S)$, $\mathcal{H}(\widetilde{K}_S)$ and $\mathcal{H}(\widehat{K}_S)$ (called *de Branges-Rovnyak reproducing kernel Hilbert spaces*) associated with the Schur-class function S have been much studied over the years, both as an object in itself and as a tool for other types of applications. Observe that the kernel $K_S(z, \zeta)$ is analytic in $z, \overline{\zeta}$ and therefore, all functions in the associated space $\mathcal{H}(K_S)$ are analytic on $\mathbb{D}$. The kernel $\widetilde{K}_S$ is analytic in $\overline{z}$ and ζ and the associated space $\mathcal{H}(\widetilde{K}_S)$ consists of conjugate-analytic functions. Similarly, the elements of $\mathcal{H}(\widehat{K}_S)$ are the functions of the form $f = \begin{bmatrix} f_+ \\ f_- \end{bmatrix}$ where f_+ is analytic and f_- is conjugate-analytic. The special role of the de Branges-Rovnyak spaces in connection with the transfer-function realization for Schur-class functions is illustrated in the following three theorems.

Theorem 1.3. *Suppose that the function S is in the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ and let $\mathcal{H}(K_S)$ be the associated de Branges-Rovnyak space. Define operators A, B, C, D by*

$$A\colon f(z) \mapsto \frac{f(z) - f(0)}{z}, \qquad B\colon u \mapsto \frac{S(z) - S(0)}{z}u,$$
$$C\colon f(z) \mapsto f(0), \qquad D\colon u \mapsto S(0)u.$$

Then the operator colligation $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ defines a coisometry from $\mathcal{H}(K_S) \oplus \mathcal{U}$ to $\mathcal{H}(K_S) \oplus \mathcal{Y}$. Moreover, $\mathbf{U}$ is observable and its transfer function equals $S(z)$. Finally, any observable coisometric colligation $\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ with its transfer function equal to S is unitarily equivalent to $\mathbf{U}$.

Theorem 1.4. *Suppose that the function S is in the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ and let $\mathcal{H}(\widetilde{K}_S)$ be the associated dual de Branges-Rovnyak space. Define*

$$\widetilde{A}\colon g(z) \mapsto \overline{z}g(z) - S(z)^*\widetilde{g}(0), \qquad \widetilde{B}\colon u \mapsto (I - S(z)^*S(0))u,$$
$$\widetilde{C}\colon g(z) \mapsto \widetilde{g}(0), \qquad \widetilde{D}\colon u \mapsto S(0)u,$$

where $\widetilde{g}(0)$ *is the unique vector in* $\mathcal{Y}$ *such that*

$$\langle \widetilde{g}(0), y\rangle_{\mathcal{Y}} = \left\langle g(z), \frac{S(z)^* - S(0)^*}{\overline{z}} y \right\rangle_{\mathcal{H}(\widetilde{K}_S)} \quad \textit{for all } y \in \mathcal{Y}.$$

Then the operator colligation $\widetilde{\mathbf{U}} = \left[\begin{smallmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{smallmatrix}\right]$ *defines an isometry from* $\mathcal{H}(\widetilde{K}_S) \oplus \mathcal{U}$ *to* $\mathcal{H}(\widetilde{K}_S) \oplus \mathcal{Y}$. *Moreover,* $\widetilde{\mathbf{U}}$ *is controllable and its transfer function equals* $S(z)$. *Finally any controllable isometric colligation* $\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ *with its transfer function equal to* S *is unitarily equivalent to* $\widetilde{\mathbf{U}}$.

Theorem 1.5. *Suppose that the function* S *is in the Schur class* $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ *and let* $\widehat{K}(z,\zeta)$ *be the positive kernel on* $\mathbb{D}$ *given by* (1.2). *Define operators* $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}$ *by*

$$\widehat{A}\colon \begin{bmatrix} f(z) \\ g(z) \end{bmatrix} \mapsto \begin{bmatrix} [f(z) - f(0)]/z \\ \overline{z}g(z) - S(z)^* f(0) \end{bmatrix}, \qquad \widehat{B}\colon u \mapsto \begin{bmatrix} \frac{S(z)-S(0)}{z} u \\ (I - S(z)^* S(0))u \end{bmatrix},$$

$$\widehat{C}\colon \begin{bmatrix} f(z) \\ g(z) \end{bmatrix} \mapsto f(0), \qquad \widehat{D}\colon u \mapsto S(0)u.$$

Then the operator colligation $\widehat{\mathbf{U}} = \left[\begin{smallmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & \widehat{D} \end{smallmatrix}\right]$ *defines a unitary operator from* $\mathcal{H}(\widehat{K}_S) \oplus \mathcal{U}$ *onto* $\mathcal{H}(\widehat{K}_S) \oplus \mathcal{Y}$. *Moreover,* $\widehat{\mathbf{U}}$ *is closely connected and its transfer function equals* $S(z)$. *Finally, any closely connected unitary colligation* $\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right] : (\mathcal{X} \oplus \mathcal{U}) \to (\mathcal{X} \oplus \mathcal{Y})$ *with its transfer function equal to* S *is unitarily equivalent to* $\widehat{\mathbf{U}}$.

Multivariable generalizations of these and many other related results have been obtained recently in the following way: let $\mathbf{Q}$ be a $p \times q$ matrix-valued polynomial

$$\mathbf{Q}(z) = \begin{bmatrix} \mathbf{q}_{11}(z) & \dots & \mathbf{q}_{1q}(z) \\ \vdots & & \vdots \\ \mathbf{q}_{p1}(z) & \dots & \mathbf{q}_{pq}(z) \end{bmatrix} : \mathbb{C}^n \to \mathbb{C}^{p \times q} \tag{1.4}$$

and let $\mathcal{D}_{\mathbf{Q}} \in \mathbb{C}^n$ be the domain defined by

$$\mathcal{D}_{\mathbf{Q}} = \{z \in \mathbb{C}^n \colon \|\mathbf{Q}(z)\| < 1\}.$$

Now we recall the *Schur-Agler class* $\mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ that consists, by definition, of $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions $S(z) = S(z_1, \dots, z_n)$ analytic on $\mathcal{D}_{\mathbf{Q}}$ and such that $\|S(T)\| \le 1$ for any collection of n commuting operators $T = (T_1, \dots, T_n)$ on a Hilbert space $\mathcal{K}$, subject to $\|\mathbf{Q}(T)\| < 1$. By [3, Lemma 1], the Taylor joint spectrum of the commuting n-tuple $T = (T_1, \dots, T_n)$ is contained in $\mathcal{D}_{\mathbf{Q}}$ whenever $\|\mathbf{Q}(T)\| < 1$, and hence $S(T)$ is well defined by the Taylor functional calculus for any $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function S which is analytic on $\mathcal{D}_{\mathbf{Q}}$. The following result appears in [2, 6] (see also [3] for the scalar-valued case $\mathcal{U} = \mathcal{Y} = \mathbb{C}$) and is a multivariable analog of Theorem 1.1. Here and in what follows, we use notation $\mathcal{H}^n := \oplus_1^n \mathcal{H}$ for a Hilbert space $\mathcal{H}$. Also we will often abuse notation and will write $\mathbf{Q}(z)$ instead of $\mathbf{Q}(z) \otimes I_{\mathcal{H}}$.

Theorem 1.6. *Let S be a $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function defined on $\mathcal{D}_{\mathbf{Q}}$. The following statements are equivalent:*

(1) *S belongs to $\mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$.*

(2a) *There exists a positive kernel*

$$\mathbb{K}_L = \begin{bmatrix} K_{11}^L & \dots & K_{1p}^L \\ \vdots & & \vdots \\ K_{p1}^L & \dots & K_{pp}^L \end{bmatrix} : \mathcal{D}_{\mathbf{Q}} \times \mathcal{D}_{\mathbf{Q}} \to \mathcal{L}(\mathcal{Y}^p) \tag{1.5}$$

such that for every $z, \zeta \in \mathcal{D}_{\mathbf{Q}}$,

$$I_{\mathcal{Y}} - S(z)S(\zeta)^* = \sum_{j=1}^{p} K_{j,j}^L(z,\zeta) - \sum_{k=1}^{q} \sum_{i,\ell=1}^{p} \mathbf{q}_{ik}(z)\overline{\mathbf{q}_{\ell k}(\zeta)} K_{i,\ell}^L(z,\zeta). \tag{1.6}$$

(2b) *There exists a positive kernel*

$$\mathbb{K}_R = \begin{bmatrix} K_{11}^R & \dots & K_{1q}^R \\ \vdots & & \vdots \\ K_{q1}^R & \dots & K_{qq}^R \end{bmatrix} : \mathcal{D}_{\mathbf{Q}} \times \mathcal{D}_{\mathbf{Q}} \to \mathcal{L}(\mathcal{U}^q) \tag{1.7}$$

so that for every $z, \zeta \in \mathcal{D}_{\mathbf{Q}}$,

$$I_{\mathcal{U}} - S(z)^*S(\zeta) = \sum_{k=1}^{q} K_{k,k}^R(z,\zeta) - \sum_{j=1}^{p} \sum_{i,\ell=1}^{q} \overline{\mathbf{q}_{ji}(z)}\mathbf{q}_{j\ell}(\zeta) K_{i,\ell}^R(z,\zeta). \tag{1.8}$$

(2c) *There exist kernels $\mathbb{K}_L$ and $\mathbb{K}_R$ of the form* (1.5), (1.7) *and satisfying identities* (1.6), (1.8), *and a kernel*

$$\mathbb{K}_{LR} = \begin{bmatrix} K_{1,1}^{LR} & \dots & K_{1,q}^{LR} \\ \vdots & & \vdots \\ K_{p,1}^{LR} & \dots & K_{p,q}^{LR} \end{bmatrix} : \mathcal{D}_{\mathbf{Q}} \times \mathcal{D}_{\mathbf{Q}} \mapsto \mathcal{L}(\mathcal{U}^q, \mathcal{Y}^p) \tag{1.9}$$

satisfying

$$S(z) - S(\zeta) = \sum_{i=1}^{p} \sum_{\ell=1}^{q} (\mathbf{q}_{i\ell}(z) - \mathbf{q}_{i\ell}(\zeta)) K_{i,\ell}^{LR}(z,\zeta) \tag{1.10}$$

and such that the kernel

$$\mathbb{K}(z,\zeta) = \begin{bmatrix} \mathbb{K}_L(z,\zeta) & \mathbb{K}_{LR}(z,\zeta) \\ \mathbb{K}_{RL}(z,\zeta) & \mathbb{K}_R(z,\zeta) \end{bmatrix} \tag{1.11}$$

is positive on $\mathcal{D}_{\mathbf{Q}} \times \mathcal{D}_{\mathbf{Q}}$, where $\mathbb{K}_{RL}(z,\zeta) := \mathbb{K}_{LR}(\zeta, z)^*$.

(3) *There exist an auxiliary Hilbert space* $\mathcal{X}$ *and a unitary connecting operator* (*or colligation*) $\mathbf{U}$ *of the structured form*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_{11} & \dots & A_{1p} & B_1 \\ \vdots & & \vdots & \vdots \\ A_{q1} & \dots & A_{qp} & B_q \\ C_1 & \dots & C_p & D \end{bmatrix} : \begin{bmatrix} \mathcal{X}^p \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^q \\ \mathcal{Y} \end{bmatrix} \tag{1.12}$$

so that $S(z)$ *can be realized in the form*

$$S(z) = D + C\left(I_{\mathcal{X}^p} - (\mathbf{Q}(z) \otimes I_{\mathcal{X}})A\right)^{-1} (\mathbf{Q}(z) \otimes I_{\mathcal{X}})B \quad (z \in \mathcal{D}_{\mathbf{Q}}). \tag{1.13}$$

(4) $S(z)$ *has a realization as in* (1.13) *where the connecting operator* $\mathbf{U}$ *is any one of* (i) *isometric,* (ii) *coisometric, or* (iii) *contractive.*

In what follows, formulas (1.6) and (1.8) will be called respectively a *left* and a *right* Agler decomposition respectively.

The objective of this paper is to extend Theorems 1.3, 1.4 and 1.5 to the present multivariable setting. In other words we are aiming at constructing realizations for a given Schur-Agler function $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ satisfying certain metric properties (such as being coisometric, isometric, or unitary) in a certain canonical way. The latter means in particular, that we will point out a canonical choice of the state spaces for these realizations and that under certain minimality conditions, the constructed realizations will be unique up to unitary equivalence.

We say that a colligation $\mathbf{U}$ of the form (1.12) is *unitarily equivalent* to a colligation

$$\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & D \end{bmatrix} : \begin{bmatrix} \widetilde{\mathcal{X}}^p \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \widetilde{\mathcal{X}}^q \\ \mathcal{Y} \end{bmatrix} \tag{1.14}$$

if there exists a unitary operator $U \colon \mathcal{X} \to \widetilde{\mathcal{X}}$ such that

$$\begin{bmatrix} \oplus_{i=1}^q U & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & D \end{bmatrix} \begin{bmatrix} \oplus_{i=1}^p U & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}. \tag{1.15}$$

Equality (1.15) is what we need to guarantee (as in the univariate case) that the transfer functions of two unitarily equivalent colligations coincide. We next extend Definition 1.2 to the present setting. We denote by $\mathcal{I}_{n,i} \colon \mathcal{X} \to \mathcal{X}^n$ the inclusion map of the space $\mathcal{X}$ into the ith slot in the direct-sum space $\mathcal{X}^d = \bigoplus_{k=1}^d \mathcal{X}$; the adjoint then is the orthogonal projection of $\mathcal{X}^n$ down to the ith component:

$$\mathcal{I}_{n,i} \colon x_i \mapsto \begin{bmatrix} 0 \\ \vdots \\ x_i \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad \mathcal{I}_{n,i}^* \colon \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} \mapsto x_i. \tag{1.16}$$

We will say that the output pair (C, A) is **Q**-*observable* if the observability subspace $\mathcal{H}^{\mathcal{O}}_{C,A}$ defined below is equal to $\mathcal{X}$:

$$\mathcal{H}^{\mathcal{O}}_{C,A} := \bigvee \left\{\mathcal{I}^*_{p,j}(I_{\mathcal{X}^p} - A^*\mathbf{Q}(z)^*)^{-1}C^*y: \ z \in \mathcal{D}_{\mathbf{Q}}, \ y \in \mathcal{Y}, \ j = 1, \dots, p\right\} = \mathcal{X}.$$

We will say that the input pair (A, B) is **Q**-*controllable* if the controllability subspace $\mathcal{H}^{\mathcal{C}}_{A,B}$ defined below is equal to $\mathcal{X}$:

$$\mathcal{H}^{\mathcal{C}}_{A,B} := \bigvee \left\{\mathcal{I}^*_{q,k}(I_{\mathcal{X}^q} - A\mathbf{Q}(z))^{-1}Bu: \ z \in \mathcal{D}_{\mathbf{Q}}, \ u \in \mathcal{U}, \ k = 1, \dots, q\right\} = \mathcal{X}.$$

Definition 1.7. The structured colligation (1.12) is called **Q**-*observable* if the output pair (C, A) is **Q**-*observable*. It is called **Q**-*controllable* if the input pair (A, B) is **Q**-*controllable* and it is called *closely connected* if

$$\mathcal{H}^{\mathcal{O}}_{C,A} \bigvee \mathcal{H}^{\mathcal{C}}_{A,B} = \mathcal{X}.$$

In analogy with the univariate case, a realization of the form (1.13) is called *coisometric*, *isometric*, *unitary* or *contractive* if the operator **U** is respectively, coisometric, isometric, unitary or just contractive. It turns out that a more useful analogue of the notion of "isometric" or "coisometric" realization appearing in the classical univariate case is not that the whole connecting operator **U** (or $\mathbf{U}^*$) be isometric, but rather that **U** (respectively, $\mathbf{U}^*$) be isometric on a certain canonical subspace of $\mathcal{X}^p \oplus \mathcal{U}$ (of $\mathcal{X}^q \oplus \mathcal{Y}$, respectively).

Definition 1.8. The colligation **U** of the form (1.12) is called

1) *weakly coisometric* if the adjoint $\mathbf{U}^* : \mathcal{X}^q \oplus \mathcal{Y} \to \mathcal{X}^p \oplus \mathcal{U}$ is contractive and isometric on the subspace

$$\mathcal{D}_{\mathbf{U}^*} := \bigvee_{\zeta \in \mathcal{D}_{\mathbf{Q}},\, y \in \mathcal{Y}} \begin{bmatrix} \mathbf{Q}(\zeta)^*(I - A^*\mathbf{Q}(\zeta)^*)^{-1}C^*y \\ y \end{bmatrix} \subset \begin{bmatrix} \mathcal{X}^q \\ \mathcal{Y} \end{bmatrix}; \tag{1.17}$$

2) *weakly isometric* if **U** is contractive and isometric on the subspace

$$\widetilde{\mathcal{D}}_{\mathbf{U}} := \bigvee_{\zeta \in \mathcal{D}_{\mathbf{Q}},\, u \in \mathcal{U}} \begin{bmatrix} \mathbf{Q}(\zeta)(I - A\mathbf{Q}(\zeta))^{-1}Bu \\ u \end{bmatrix} \subset \begin{bmatrix} \mathcal{X}^p \\ \mathcal{U} \end{bmatrix}; \tag{1.18}$$

3) *weakly unitary* if it is weakly isometric and weakly coisometric.

The paper is organized as follows. After the present Introduction, Section 2 sets up some matricial notation which helps to streamline the computations to come and also explains why there is no loss of generality in assuming that $0 \in \mathcal{D}_{\mathbf{Q}}$ and $\mathbf{Q}(0) = 0$. Sections 3 and 4 review needed material from [7] concerning weakly coisometric and weakly isometric canonical functional models respectively and the corresponding respective analogues of Theorems 1.3 and 1.4 to the multivariable **Q**-setting. Section 5 introduces two-component canonical functional models associated with a matrix polynomial **Q** and obtains the analogue of Theorem 1.5 in complete detail and generality.

We mention that special cases of the formalism here corresponding to $\mathbf{Q}$ equal to a row linear matrix polynomial of the form $\mathbf{Q}(z) = \begin{bmatrix} z_1 & \cdots z_d \end{bmatrix}$ or $\mathbf{Q}$ equal to a diagonal linear matrix polynomial $\mathbf{Q}(z) = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_d \end{bmatrix}$ lead to more familiar function theory over the unit ball $\mathbb{B}^d = \{z = (z_1, \dots, z_d) \colon \sum_{j=1}^d |z_j|^2 < 1\}$ and the unit polydisk $\mathbb{D}^d = \{z = (z_1, \dots, z_d) \colon |z_j| < 1 \text{ for } j = 1, \dots, d\}$ respectively; these specializations are explained in [7]. We present the special details and applications of the two-component canonical functional model colligations for these two special cases in separate articles [8, 9].

2. Preliminaries

In this section we collect some preliminaries needed for the subsequent analysis. We first remark that realization formulas become much simpler if there exists a point $z_0 \in \mathcal{D}_{\mathbf{Q}}$ such that $\mathbf{Q}(z_0) = 0$. In general, such a point may not exist. However, as the next lemma shows, we may assume without loss of generality that such a point does exist.

Lemma 2.1. *Let $z_0 \in \mathcal{D}_{\mathbf{Q}}$ so that the matrix $Q_0 = \mathbf{Q}(z_0)$ is a strictly contractive matrix. Then*

1. *The $p \times q$ matrix-valued function*
$$\widehat{\mathbf{Q}}(z) = (I_p - Q_0 Q_0^*)^{-\frac{1}{2}} (\mathbf{Q}(z) - Q_0) (I_q - Q_0^* \mathbf{Q}(z))^{-1} (I_q - Q_0^* Q_0)^{\frac{1}{2}} \tag{2.1}$$
is the transfer function of the unitary colligation
$$\mathbf{U}_0 = \begin{bmatrix} Q_0^* & (I_q - Q_0^* Q_0)^{\frac{1}{2}} \\ (I_p - Q_0 Q_0^*)^{\frac{1}{2}} & -Q_0 \end{bmatrix}. \tag{2.2}$$
2. *The domains $\mathcal{D}_{\mathbf{Q}}$ and $\mathcal{D}_{\widehat{\mathbf{Q}}}$ coincide.*
3. *The classes $\mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ and $\mathcal{SA}_{\widehat{\mathbf{Q}}}(\mathcal{U}, \mathcal{Y})$ are the same.*

Proof. Write (2.1) as
$$\begin{aligned} \widehat{\mathbf{Q}}(z) = & - (I_p - Q_0 Q_0^*)^{-\frac{1}{2}} Q_0 (I_q - Q_0^* Q_0)^{\frac{1}{2}} \\ & + (I_p - Q_0 Q_0^*)^{\frac{1}{2}} \mathbf{Q}(z) (I_q - Q_0^* \mathbf{Q}(z))^{-1} (I_q - Q_0^* Q_0)^{\frac{1}{2}} \end{aligned}$$
and use self-evident equalities
$$\begin{aligned} (I_p - Q_0 Q_0^*)^{-\frac{1}{2}} Q_0 (I_p - Q_0^* Q_0)^{\frac{1}{2}} &= Q_0, \\ \mathbf{Q}(z) (I_q - Q_0^* \mathbf{Q}(z))^{-1} &= (I_p - \mathbf{Q}(z) Q_0^*)^{-1} \mathbf{Q}(z) \end{aligned}$$
to get
$$\widehat{\mathbf{Q}}(z) = -Q_0 + (I_p - Q_0 Q_0^*)^{\frac{1}{2}} (I_p - \mathbf{Q}(z) Q_0^*)^{-1} \mathbf{Q}(z) (I_q - Q_0^* Q_0)^{\frac{1}{2}}$$

which means that $\widehat{\mathbf{Q}}(z)$ is the transfer function of the $\mathbf{Q}$-colligation $\mathbf{U}_0$ defined in (2.2). Solving (2.1) for $\mathbf{Q}$ gives

$$\begin{aligned}\mathbf{Q}(z) &= (I_p - Q_0Q_0^*)^{\frac{1}{2}}\left(I_p + \widehat{\mathbf{Q}}(z)Q_0^*\right)^{-1}\left(\widehat{\mathbf{Q}}(z) + Q_0\right)(I_q - Q_0^*Q_0)^{-\frac{1}{2}} \\ &= Q_0 + (I_p - Q_0Q_0^*)^{\frac{1}{2}}\left(I_q + \widehat{\mathbf{Q}}(z)Q_0^*\right)^{-1}\widehat{\mathbf{Q}}(z)\,(I_q - Q_0^*Q_0)^{\frac{1}{2}}\end{aligned} \tag{2.3}$$

and thus $\mathbf{Q}(z)$ is the transfer function of the $\widehat{\mathbf{Q}}$-colligation

$$\widehat{\mathbf{U}}_0 = \begin{bmatrix} -Q_0^* & (I_q - Q_0^*Q_0)^{\frac{1}{2}} \\ (I_p - Q_0Q_0^*)^{\frac{1}{2}} & Q_0 \end{bmatrix}.$$

It is readily seen that $\mathbf{U}_0$ and $\widehat{\mathbf{U}}_0$ are unitary and therefore

$$\begin{aligned} I_p - \widehat{\mathbf{Q}}(z)\widehat{\mathbf{Q}}(\zeta)^* &= (I_p - Q_0Q_0^*)^{\frac{1}{2}}\,(I_p - \mathbf{Q}(z)Q_0^*)^{-1}\,(I_p - \mathbf{Q}(z)\mathbf{Q}(\zeta)^*) \\ &\quad \times (I_p - Q_0\mathbf{Q}(\zeta)^*)^{-1}\,(I_p - Q_0Q_0^*)^{\frac{1}{2}}, \end{aligned}$$

from which we conclude that $\|\widehat{\mathbf{Q}}(z)\| < 1$ if and only if $\|\mathbf{Q}(z)\| < 1$ which proves the statement (2) of the lemma.

To prove statement (3), we take an arbitrary function $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ which admits a unitary realization (1.13) by Theorem 1.6. Substituting (2.3) into (1.13), leads us to

$$S(z) = \widehat{D} + \widehat{C}\left(I_{\mathcal{X}^p} - \widehat{\mathbf{Q}}(z)\widehat{A}\right)^{-1}\widehat{\mathbf{Q}}(z)\widehat{B} \tag{2.4}$$

where

$$\begin{aligned} \widehat{A} &= (I - Q_0^*Q_0)^{-\frac{1}{2}}\,(A - Q_0^*)\,(I - Q_0A)^{-1}\,(I - Q_0Q_0^*)^{\frac{1}{2}}, \\ \widehat{B} &= (I - Q_0^*Q_0)^{\frac{1}{2}}\,(I - AQ_0)^{-1}\,B, \\ \widehat{C} &= C\,(I - Q_0A)^{-1}\,(I - Q_0^*Q_0)^{\frac{1}{2}}, \\ \widehat{D} &= D + C\,(I - Q_0A)^{-1}\,Q_0B = S(z_0). \end{aligned} \tag{2.5}$$

Representation (2.4) means that S is the transfer function of the $\widehat{\mathbf{Q}}$-colligation

$$\widehat{\mathbf{U}} = \begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & \widehat{D} \end{bmatrix} : \begin{bmatrix} \mathcal{X}^p \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^q \\ \mathcal{Y} \end{bmatrix}.$$

The verification of the fact that the colligation $\widehat{\mathbf{U}}$ is unitary is straightforward; it involves the explicit formulas for its block entries and the fact that the operators $\mathbf{U}$ and $\mathbf{U}_0$ are unitary. Since the $\widehat{\mathbf{Q}}$-colligation $\widehat{\mathbf{U}}$ is unitary, its transfer function S belongs to $\mathcal{SA}_{\widehat{\mathbf{Q}}}(\mathcal{U}, \mathcal{Y})$, by Theorem 1.3. Thus, we verified the inclusion $\mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y}) \subseteq \mathcal{SA}_{\widehat{\mathbf{Q}}}(\mathcal{U}, \mathcal{Y})$. To check the converse inclusion, we start with a function $S \in \mathcal{SA}_{\widehat{\mathbf{Q}}}(\mathcal{U}, \mathcal{Y})$ and substitute (2.1) into its unitary $\widehat{\mathbf{Q}}$-realization (2.4). Straightforward calculations show that S is of the form (1.13) with the operators A, B, C and D uniquely recovered from the system (2.5). By the preceding arguments, the operator $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is unitary and then, $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$, by Theorem 1.6. □

Remark 2.2. This trick exhibited in Lemma 2.1 of reducing to the case where $S(0) = 0$ comes up in the context of control theory and there is known as *loop-shifting* (see Exercise 8.11 page 277 in [13]).

From now on, we assume that

$$0 \in \mathcal{D}_{\mathbf{Q}} \quad \text{and} \quad \mathbf{Q}(0) = 0. \tag{2.6}$$

We next represent identities (1.6), (1.8) and (1.10) in a more matricial form. In what follows, $\{\mathbf{e}_1, \dots, \mathbf{e}_p\}$ and $\{\widetilde{\mathbf{e}}_1, \dots, \widetilde{\mathbf{e}}_q\}$ will stand for the standard bases for $\mathbb{C}^p$ and $\mathbb{C}^q$ respectively. Let us define the operator-valued polynomials

$$M_j(z) = \begin{bmatrix} M_j^L & 0 \\ 0 & M_j^R(z) \end{bmatrix} \quad \text{and} \quad N_k(z) = \begin{bmatrix} N_k^L(z) & 0 \\ 0 & N_k^R \end{bmatrix} \tag{2.7}$$

for $j = 1, \dots, p$ and $k = 1, \dots, q$, where

$$M_j^L = \mathbf{e}_j \otimes I_{\mathcal{Y}}, \qquad N_k^R = \widetilde{\mathbf{e}}_k \otimes I_{\mathcal{U}}, \tag{2.8}$$

$$M_j^R(z) = (\mathbf{Q}^\top(z)\mathbf{e}_j) \otimes I_{\mathcal{U}} = \begin{bmatrix} \mathbf{q}_{j1}(z) I_{\mathcal{U}} \\ \vdots \\ \mathbf{q}_{jq}(z) I_{\mathcal{U}} \end{bmatrix} = \sum_{k=1}^{q} \mathbf{q}_{jk}(z) N_k^R, \tag{2.9}$$

$$N_k^L(z) = (\overline{\mathbf{Q}(z)}\widetilde{\mathbf{e}}_k) \otimes I_{\mathcal{Y}} = \begin{bmatrix} \overline{\mathbf{q}_{1k}(z)} I_{\mathcal{Y}} \\ \vdots \\ \overline{\mathbf{q}_{pk}(z)} I_{\mathcal{Y}} \end{bmatrix} = \sum_{j=1}^{p} \overline{\mathbf{q}_{jk}(z)} M_j^L. \tag{2.10}$$

Equalities (1.6), (1.8) and (1.10) can be written in terms of the notation (2.8)–(2.10) as

$$I_{\mathcal{Y}} - S(z)S(\zeta)^* = \sum_{j=1}^{p} M_j^{L*} \mathbb{K}_L(z,\zeta) M_j^L - \sum_{k=1}^{q} N_k^L(z)^* \mathbb{K}_L(z,\zeta) N_k^L(\zeta), \tag{2.11}$$

$$I_{\mathcal{U}} - S(z)^* S(\zeta) = \sum_{k=1}^{q} N_k^{R*} \mathbb{K}_R(z,\zeta) N_k^R - \sum_{j=1}^{p} M_j^R(z)^* \mathbb{K}_R(z,\zeta) M_j^R(\zeta), \tag{2.12}$$

$$S(z) - S(\zeta) = \sum_{k=1}^{q} N_k^L(z)^* \mathbb{K}_{LR}(z,\zeta) N_k^R - \sum_{j=1}^{p} M_j^{L*} \mathbb{K}_{LR}(z,\zeta) M_j^R(\zeta). \tag{2.13}$$

Taking adjoints in (2.13) and switching z and ζ we get

$$S(z)^* - S(\zeta)^* = \sum_{j=1}^{p} M_j^R(z)^* \mathbb{K}_{RL}(z,\zeta) M_j^L - \sum_{k=1}^{q} N_k^{R*} \mathbb{K}_{RL}(z,\zeta) N_k^L(\zeta) \tag{2.14}$$

and observe that the latter four identities are equivalent to the single block matrix identity

$$\begin{bmatrix} I_{\mathcal{Y}} \\ S(z)^* \end{bmatrix} \begin{bmatrix} I_{\mathcal{Y}} & S(\zeta) \end{bmatrix} - \begin{bmatrix} S(z) \\ I_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} S(\zeta)^* & I_{\mathcal{U}} \end{bmatrix} \tag{2.15}$$
$$= \sum_{j=1}^{p} M_j(z)^* \mathbb{K}(z,\zeta) M_j(\zeta) - \sum_{k=1}^{q} N_k(z)^* \mathbb{K}(z,\zeta) N_k(\zeta),$$

where $\mathbb{K}$ is the kernel of the form (1.11). We will also refer to equalities (2.11) and (2.12) (which are the same as (1.6) and (1.8)) as left and right Agler decompositions for S respectively, while the equality (2.15) will be referred to simply as an Agler decomposition.

3. Weakly coisometric realizations

For every function $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ with a fixed left Agler decomposition (1.6), one can construct a weakly coisometric realization in a certain canonical way. This was shown in [7] and now will be recalled. For functions $f \in \mathcal{H}(\mathbb{K}_L)^p$, we use the notation

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_p \end{bmatrix} \quad \text{where} \quad f_k = \begin{bmatrix} f_{k,1} \\ \vdots \\ f_{k,p} \end{bmatrix} \in \mathcal{H}(\mathbb{K}_L).$$

We say that the operator $A\colon \mathcal{H}(\mathbb{K}_L)^p \to \mathcal{H}(\mathbb{K}_L)^q$ *solves the* $\mathbf{Q}$*-coupled Gleason problem for* $\mathcal{H}(\mathbb{K}_L)$ if

$$\sum_{k=1}^{p} (f_{k,k}(z) - f_{k,k}(0)) = \sum_{j=1}^{p}\sum_{k=1}^{q} \mathbf{q}_{jk}(z) \left[Af\right]_{k,j}(z) = \sum_{j=1}^{p} \left[\mathbf{Q}(z)Af\right]_{j,j}(z) \tag{3.1}$$

for all $f \in \mathcal{H}(\mathbb{K}_L)^p$. Observe that the second equality in (3.1) is a tautology and its verification relies just on the definition of matrix multiplication. Similarly, we say that the operator $B\colon \mathcal{U} \to \mathcal{H}(\mathbb{K}_L)^q$ *solves the* $\mathbf{Q}$*-coupled* $\mathcal{H}(\mathbb{K})$*-Gleason problem for* S if the identity

$$S(z)u - S(0)u = \sum_{j=1}^{p}\sum_{k=1}^{q} \mathbf{q}_{jk}(z)[Bu]_{k,j}(z) = \sum_{j=1}^{p} \left[\mathbf{Q}(z)Bu\right]_{j,j}(z) \tag{3.2}$$

holds for all $u \in \mathcal{U}$, where similarly to (3.1), the second equality is a tautology.

Definition 3.1. We say that the operator-block matrix

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}(\mathbb{K}_L)^p \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(\mathbb{K}_L)^q \\ \mathcal{Y} \end{bmatrix} \tag{3.3}$$

is a *canonical functional-model (*abbreviated to **c.f.m.** in what follows*) colligation* for the given function $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ with the left Agler decomposition $\mathbb{K}_L$ if

1. $\mathbf{U}$ is contractive.
2. The operator A solves the $\mathbf{Q}$-coupled Gleason problem (3.1).

3. The operator B solves the $\mathbf{Q}$-coupled Gleason problem (3.2) for S.
4. The operators $C : \mathcal{H}(\mathbb{K}_L)^p \to \mathcal{Y}$ and $D : \mathcal{U} \to \mathcal{Y}$ are given by

$$C\colon \begin{bmatrix} f_1(z) \\ \vdots \\ f_p(z) \end{bmatrix} \mapsto f_{1,1}(0) + \cdots + f_{p,p}(0), \quad D\colon u \mapsto S(0)u.$$

With a given left Agler decomposition $\mathbb{K}_L$ of a function $S \in \mathcal{S}A_{\mathbf{Q}}(\mathcal{U},\mathcal{Y})$, we associate the kernel

$$\mathbb{T}^L(z,\zeta) := \begin{bmatrix} \mathbb{K}_L(z,\zeta)M_1^L \\ \vdots \\ \mathbb{K}_L(z,\zeta)M_p^L \end{bmatrix} :\ \mathcal{D}_{\mathbf{Q}} \times \mathcal{D}_{\mathbf{Q}} \to \mathcal{L}(\mathcal{Y},\mathcal{Y}^{p^2}). \tag{3.4}$$

It is not hard to check using the reproducing kernel property, the definition (3.4) of $\mathbb{T}^L$ and equality (2.10), that for every $z,\zeta \in \mathcal{D}_{\mathbf{Q}}$ and $y,y' \in \mathcal{Y}$,

$$\left\langle \mathbb{T}^L(\cdot,\zeta)y,\, \mathbb{T}^L(\cdot,z)y' \right\rangle_{\mathcal{H}(\mathbb{K})^p} = \sum_{j=1}^p \left\langle M_j^{L*}\mathbb{K}_L(z,\zeta)M_j^L y,\, y' \right\rangle_{\mathcal{Y}},$$

$$\left\langle \mathbf{Q}(\zeta)^*\mathbb{T}^L(\cdot,\zeta)y,\, \mathbf{Q}(z)^*\mathbb{T}^L(\cdot,z)y' \right\rangle_{\mathcal{H}(\mathbb{K})^q} = \sum_{k=1}^q \left\langle N_k^L(z)^*\mathbb{K}_L(z,\zeta)N_k^L(\zeta)y,\, y' \right\rangle_{\mathcal{Y}}.$$

Making use of the two latter equalities, one can write the identity

$$\sum_{k=1}^q N_k^L(z)^*\mathbb{K}_L(z,\zeta)N_k^L(\zeta) + I_{\mathcal{Y}} = \sum_{j=1}^p M_j^{L*}\mathbb{K}_L(z,\zeta)M_j^L + S(z)S(\zeta)^*$$

(which is just a rearrangement of the left Agler decomposition (2.11)) in the inner product form as

$$\left\langle \begin{bmatrix} \mathbf{Q}(\zeta)^*\mathbb{T}^L(\cdot,\zeta)y \\ y \end{bmatrix}, \begin{bmatrix} \mathbf{Q}(z)^*\mathbb{T}^L(\cdot,z)y' \\ y' \end{bmatrix} \right\rangle_{\mathcal{H}(\mathbb{K}_L)^q \oplus \mathcal{Y}}$$
$$= \left\langle \begin{bmatrix} \mathbb{T}^L(\cdot,\zeta)y \\ S(\zeta)^*y \end{bmatrix}, \begin{bmatrix} \mathbb{T}^L(\cdot,z)y' \\ S(z)^*y' \end{bmatrix} \right\rangle_{\mathcal{H}(\mathbb{K}_L)^p \oplus \mathcal{U}}.$$

Therefore the linear map

$$V\colon \begin{bmatrix} \mathbf{Q}(\zeta)^*\mathbb{T}^L(\cdot,\zeta)y \\ y \end{bmatrix} \to \begin{bmatrix} \mathbb{T}^L(\cdot,\zeta)y \\ S(\zeta)^*y \end{bmatrix} \tag{3.5}$$

extends to the isometry from

$$\mathcal{D}_V = \bigvee_{\zeta \in \mathcal{D}_{\mathbf{Q}},\, y \in \mathcal{Y}} \begin{bmatrix} \mathbf{Q}(\zeta)^*\mathbb{T}^L(\cdot,\zeta)y \\ y \end{bmatrix} \subset \begin{bmatrix} \mathcal{H}(\mathbb{K}_L)^q \\ \mathcal{Y} \end{bmatrix}$$

onto

$$\mathcal{R}_V = \bigvee_{\zeta \in \mathcal{D}_{\mathbf{Q}},\, y \in \mathcal{Y}} \begin{bmatrix} \mathbb{T}^L(\cdot,\zeta)y \\ S(\zeta)^*y \end{bmatrix} \subset \begin{bmatrix} \mathcal{H}(\mathbb{K}_L)^p \\ \mathcal{U} \end{bmatrix}.$$

Due to condition (2.6), $\mathcal{D}_V$ splits into a direct sum $\mathcal{D}_V = \mathcal{D} \oplus \mathcal{Y}$ where

$$\mathcal{D} = \bigvee_{\zeta \in \mathcal{D}_{\mathbf{Q}},\, y \in \mathcal{Y}} \mathbf{Q}(\zeta)^* \mathbb{T}^L(\cdot, \zeta) y \subset \mathcal{H}(\mathbb{K}_L)^q. \tag{3.6}$$

Furthermore, the defect spaces

$$\mathcal{D}_V^\perp := (\mathcal{H}(\mathbb{K}_L)^q \oplus \mathcal{Y}) \ominus \mathcal{D}_V \cong \mathcal{D}^\perp \quad \text{and} \quad \mathcal{R}_V^\perp := (\mathcal{H}(\mathbb{K}_L)^p \oplus \mathcal{U}) \ominus \mathcal{R}_V$$

can be characterized as

$$\mathcal{D}^\perp = \left\{ f \in \mathcal{H}(\mathbb{K}_L)^q : \sum_{j=1}^{q} [\mathbf{Q}f]_{j,j}(z) \equiv 0 \right\}, \tag{3.7}$$

$$\mathcal{R}_V^\perp = \left\{ \begin{bmatrix} f \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(\mathbb{K}_L)^p \\ \mathcal{U} \end{bmatrix} : \sum_{j=1}^{p} f_{j,j}(z) + S(z)u \equiv 0 \right\}. \tag{3.8}$$

The following two results were proved in [7].

Theorem 3.2. *Given a left Agler decomposition $\mathbb{K}_L$ for a function $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$, let V be the isometric operator associated with this decomposition as in* (3.5). *A block-operator matrix* $\mathbf{U}$ *of the form* (3.3) *is a* **c.f.m.** *colligation associated with* $\mathbb{K}_L$ *if and only if* $\mathbf{U}^*$ *is a contractive extension of* V *to all of* $\mathcal{H}(\mathbb{K}_L)^q \oplus \mathcal{Y}$, *i.e.,*

$$\mathbf{U}^*|_{\mathcal{D}\oplus\mathcal{Y}} = V \quad \text{and} \quad \|\mathbf{U}^*\| \le 1. \tag{3.9}$$

Theorem 3.3. *Let S be a function in the Schur-Agler class $\mathcal{S}A_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ with given left Agler decomposition $\mathbb{K}_L$. Then*

1. *There exists a* **c.f.m.** *colligation* $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *associated with* $\mathbb{K}_L$.
2. *Every* **c.f.m.** *colligation* $\mathbf{U}$ *associated with* $\mathbb{K}_L$ *is weakly coisometric and observable and furthermore,* $S(z) = D + C(I - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)B$.
3. *Any observable weakly coisometric colligation* $\mathbf{U}'$ *of the form* (1.14) *with the transfer function equal S is unitarily equivalent to some* **c.f.m.** *colligation* $\mathbf{U}$ *for S.*

4. Weakly isometric realizations

The results concerning weakly isometric colligations associated with a given right Agler decompositions (1.8) of a function $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ are parallel to the results from the previous section and can be established in much the same way. We present them here without proofs.

Assume we are given a function $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ with a fixed right Agler decompositions $\mathbb{K}_R$. Let us introduce the kernel

$$\mathbb{T}^R(z, \zeta) := \begin{bmatrix} \mathbb{K}_R(z,\zeta) N_1^R \\ \vdots \\ \mathbb{K}_R(z,\zeta) N_q^R \end{bmatrix} : \ \mathcal{D}_{\mathbf{Q}} \times \mathcal{D}_{\mathbf{Q}} \to \mathcal{L}(\mathcal{U}, \mathcal{U}^{q^2}), \tag{4.1}$$

where $N_1^R, \dots, N_q^R$ are given in (2.8) and let

$$\widetilde{\mathcal{D}} = \bigvee_{\zeta \in \mathcal{D}_{\mathbf{Q}},\, u \in \mathcal{U}} \mathbf{Q}(\zeta)\mathbb{T}^R(\cdot,\zeta)u \subset \mathcal{H}(\mathbb{K}_R)^p. \tag{4.2}$$

Definition 4.1. Given a function $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U},\mathcal{Y})$, we shall say that the block-operator matrix

$$\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} : \begin{bmatrix} \mathcal{H}(\mathbb{K}_R)^p \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(\mathbb{K}_R)^q \\ \mathcal{Y} \end{bmatrix} \tag{4.3}$$

is a *dual canonical functional-model* (abbreviated to **d.c.f.m.** in what follows) *colligation* associated with right Agler decomposition $\mathbb{K}_R$ for S if

1. $\widetilde{\mathbf{U}}$ is contractive.
2. The restrictions of operators A and C to the subspace $\widetilde{\mathcal{D}} \subset \mathcal{H}(\mathbb{K}_R)^p$ defined in (3.3) have the following action on special kernel functions:

$$\widetilde{A}|_{\widetilde{\mathcal{D}}} : \mathbf{Q}(\zeta)\mathbb{T}^R(\cdot,\zeta)u \to \mathbb{T}^R(\cdot,\zeta)u - \mathbb{T}^R(\cdot,0)u,$$
$$\widetilde{C}|_{\widetilde{\mathcal{D}}} : \mathbf{Q}(\zeta)\mathbb{T}^R(\cdot,\zeta)u \to S(\zeta)u - S(0)u.$$

3. The operators $\widetilde{B}\colon \mathcal{U} \to \mathcal{H}(\mathbb{K}_R)^q$ and $\widetilde{D}\colon \mathcal{U} \to \mathcal{Y}$ are given by

$$\widetilde{B}\colon u \mapsto \mathbb{T}^R(\cdot,0)u, \qquad \widetilde{D}\colon u \mapsto S(0)u.$$

As in the coisometric case, one can write the rearrangement

$$\sum_{j=1}^p M_j^R(z)^*\mathbb{K}_R(z,\zeta)M_j^R(\zeta) + I_{\mathcal{U}} = \sum_{k=1}^q N_k^{R*}\mathbb{K}_R(z,\zeta)N_k^R + S(z)^*S(\zeta)$$

of the right Agler decomposition (2.12) in the inner product form as

$$\left\langle \begin{bmatrix} \mathbf{Q}(\zeta)\mathbb{T}^R(\cdot,\zeta)u \\ u \end{bmatrix}, \begin{bmatrix} \mathbf{Q}(z)\mathbb{T}^R(\cdot,z)u' \\ u' \end{bmatrix} \right\rangle_{\mathcal{H}(\mathbb{K}_R)^p\oplus\mathcal{U}}$$
$$= \left\langle \begin{bmatrix} \mathbb{T}^R(\cdot,\zeta)u \\ S(\zeta)u \end{bmatrix}, \begin{bmatrix} \mathbb{T}^R(\cdot,z)u' \\ S(z)u' \end{bmatrix} \right\rangle_{\mathcal{H}(\mathbb{K}_R)^q\oplus\mathcal{Y}}$$

to conclude then that the linear map

$$\widetilde{V}\colon \begin{bmatrix} \mathbf{Q}(\zeta)\mathbb{T}^R(\cdot,\zeta)u \\ u \end{bmatrix} \to \begin{bmatrix} \mathbb{T}^R(\cdot,\zeta)u \\ S(\zeta)u \end{bmatrix} \tag{4.4}$$

extends by continuity to define the isometry $\widetilde{V} : \mathcal{D}_{\widetilde{V}} \to \mathcal{R}_{\widetilde{V}}$ where

$$\mathcal{D}_{\widetilde{V}} = \widetilde{D} \oplus \mathcal{U} \quad \text{and} \quad \mathcal{R}_{\widetilde{V}} = \bigvee_{\zeta \in \mathcal{D}_{\mathbf{Q}},\, u \in \mathcal{U}} \begin{bmatrix} \mathbb{T}^R(\cdot,\zeta)y \\ S(\zeta)u \end{bmatrix} \subset \begin{bmatrix} \mathcal{H}(\mathbb{K}_R)^q \\ \mathcal{Y} \end{bmatrix}.$$

The two following theorems are parallel to Theorems 3.2 and 3.3.

Theorem 4.2. *Given a right Agler decomposition* $\mathbb{K}_R$ *for a function* S *in the Schur-Agler class* $\mathcal{SA}_{\mathbf{Q}}(\mathcal{U},\mathcal{Y})$, *let* $\widetilde{V}$ *be the isometric operator associated with this decomposition as in* (4.4). *A block-operator matrix* $\widetilde{\mathbf{U}}$ *of the form* (4.3) *is a* **d.c.f.m.** *colligation associated with* $\mathbb{K}_R$ *if and only if* $\widetilde{\mathbf{U}}$ *is a contractive extension of* $\widetilde{V}$ *to all of* $\mathcal{H}(\mathbb{K}_R)^q \oplus \mathcal{Y}$.

Theorem 4.3. *Let* S *be a function in the Schur-Agler class* $\mathcal{S}A_d(\mathcal{U},\mathcal{Y})$ *with given right Agler decomposition* $\mathbb{K}_R$. *Then*

1. *There exists a* **d.c.f.m.** *colligation* $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *associated with* $\mathbb{K}_R$.
2. *Every* **d.c.f.m.** *colligation* $\mathbf{U}$ *associated with* $\mathbb{K}_R$ *is weakly isometric and controllable and furthermore,* $S(z) = D + C(I - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)B$.
3. *Any controllable weakly isometric colligation* $\mathbf{U}'$ *of the form* (1.14) *with transfer function equal to* S *is unitarily equivalent to some* **d.c.f.m.** *colligation* $\mathbf{U}$ *for* S.

5. Weakly unitary realizations

In this section we will study unitary realizations of an $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U},\mathcal{Y})$ associated with a fixed Agler decomposition (2.15). Following the streamlines of Section 2, we let $\mathcal{H}(\mathbb{K})$ to be the reproducing kernel Hilbert space associated with the kernel $\mathbb{K}$ from decomposition (2.15). For functions $f \in \mathcal{H}(\mathbb{K})^n$ (where in most cases, n will be equal to p or q), we will use the following representation and notation:

$$f = \bigoplus_{i=1}^{n} f_i := \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \in \mathcal{H}(\mathbb{K})^n \quad \text{where} \quad f_i = \begin{bmatrix} f_{i,+} \\ f_{i,-} \end{bmatrix} : \mathcal{D}_{\mathbf{Q}} \to \begin{bmatrix} \mathcal{Y}^p \\ \mathcal{U}^q \end{bmatrix}, \tag{5.1}$$

so that

$$f_{i,+} = \begin{bmatrix} f_{i,+,1} \\ \vdots \\ f_{i,+,p} \end{bmatrix} \ (f_{i,+,j} : \mathcal{D}_{\mathbf{Q}} \to \mathcal{Y}), \quad f_{i,-} = \begin{bmatrix} f_{i,-,1} \\ \vdots \\ f_{i,-,q} \end{bmatrix} \ (f_{i,-,k} : \mathcal{D}_{\mathbf{Q}} \to \mathcal{U}).$$

We furthermore introduce the kernels

$$\mathbb{T}(z,\zeta) := \bigoplus_{j=1}^{p} \mathbb{K}(z,\zeta) \begin{bmatrix} M_j^L \\ 0 \end{bmatrix} : \mathcal{D}_{\mathbf{Q}} \times \mathcal{D}_{\mathbf{Q}} \to \mathcal{L}(\mathcal{Y}, (\mathcal{Y}^p \oplus \mathcal{U}^q)^p), \tag{5.2}$$

$$\widetilde{\mathbb{T}}(z,\zeta) := \bigoplus_{k=1}^{q} \mathbb{K}(z,\zeta) \begin{bmatrix} 0 \\ N_k^R \end{bmatrix} : \mathcal{D}_{\mathbf{Q}} \times \mathcal{D}_{\mathbf{Q}} \to \mathcal{L}(\mathcal{U}, (\mathcal{Y}^p \oplus \mathcal{U}^q)^q) \tag{5.3}$$

where M_j^L and N_k^R are given in (2.8). We define two linear maps $\mathbf{s} : \mathcal{H}(\mathbb{K})^p \to \mathcal{H}(\mathbb{K}_L)$ and $\widetilde{\mathbf{s}} : \mathcal{H}(\mathbb{K})^q \to \mathcal{H}(\mathbb{K}_R)$ as follows:

$$\mathbf{s} : \ f = \bigoplus_{j=1}^{p} f_j \mapsto \sum_{j=1}^{p} f_{j,+,j}, \qquad \widetilde{\mathbf{s}} : \ g = \bigoplus_{k=1}^{q} g_k \mapsto \sum_{k=1}^{q} g_{k,-,k}, \tag{5.4}$$

and observe the equalities

$$\langle f,\, \mathbb{T}(\cdot,\zeta)y\rangle_{\mathcal{H}(\mathbb{K})^p} = \langle (\mathbf{s}f)(\zeta),\, y\rangle_{\mathcal{Y}}, \qquad \langle g,\, \widetilde{\mathbb{T}}(\cdot,\zeta)u\rangle_{\mathcal{H}(\mathbb{K})^q} = \langle (\widetilde{\mathbf{s}}f)(\zeta),\, u\rangle_{\mathcal{U}} \tag{5.5}$$

holding for all $\zeta \in \mathcal{D}_{\mathbf{Q}}$, $y \in \mathcal{Y}$ and $u \in \mathcal{U}$. Indeed, for a function f of the form (5.1), we have from (5.2) by the reproducing kernel property

$$\begin{aligned}\langle f,\, \mathbb{T}(\cdot,\zeta)y\rangle_{\mathcal{H}(\mathbb{K})^p} &= \sum_{j=1}^{p}\left\langle f_j,\, \mathbb{K}(\cdot,\zeta)\begin{bmatrix} M_j^L y \\ 0\end{bmatrix}\right\rangle_{\mathcal{H}(\mathbb{K})} = \sum_{j=1}^{p}\left\langle f_j(\zeta),\, \begin{bmatrix} \mathbf{e}_j\otimes y \\ 0\end{bmatrix}\right\rangle_{\mathcal{Y}\oplus\mathcal{U}} \\ &= \sum_{j=1}^{p}\langle f_{j,+}(\zeta),\, \mathbf{e}_i\otimes y\rangle_{\mathcal{Y}} = \sum_{j=1}^{p}\langle f_{j,+,j}(\zeta),\, y\rangle_{\mathcal{Y}} = \langle (\mathbf{s}f)(\zeta),\, y\rangle_{\mathcal{Y}}\end{aligned}$$

which proves the first equality in (5.5). The proof of the second is much the same.

Lemma 5.1. *Let $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ be the kernels associated with the Agler decomposition $\mathbb{K}$ (2.15) of an $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U},\mathcal{Y})$ via formulas (5.2), (5.3). Let $M_j(z)$ and $N_k(z)$ be defined as in (2.7)–(2.10). Then for every $z,\zeta \in \mathcal{D}_{\mathbf{Q}}$, $u,u' \in \mathcal{U}$ and $y,y' \in \mathcal{Y}$,*

$$\langle \mathbb{T}(\cdot,\zeta)y,\, \mathbb{T}(\cdot,z)y'\rangle_{\mathcal{H}(\mathbb{K})^p} = \sum_{j=1}^{p}\left\langle \mathbb{K}_L(z,\zeta)M_j^L y,\, M_j^L y'\right\rangle_{\mathcal{Y}^p}, \tag{5.6}$$

$$\left\langle \widetilde{\mathbb{T}}(\cdot,\zeta)u,\, \widetilde{\mathbb{T}}(\cdot,z)u'\right\rangle_{\mathcal{H}(\mathbb{K})^q} = \sum_{k=1}^{q}\left\langle \mathbb{K}_R(z,\zeta)N_k^R u,\, N_k^R u'\right\rangle_{\mathcal{U}^q}, \tag{5.7}$$

$$\langle \mathbf{Q}(\zeta)^*\mathbb{T}(\cdot,\zeta)y,\, \mathbf{Q}(z)^*\mathbb{T}(\cdot,z)y'\rangle_{\mathcal{H}(\mathbb{K})^q} = \sum_{k=1}^{q}\left\langle \mathbb{K}_L(z,\zeta)N_k^L(\zeta)y,\, N_k^L(z)y'\right\rangle_{\mathcal{Y}^p}, \tag{5.8}$$

$$\left\langle \mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u,\, \mathbf{Q}(z)\widetilde{\mathbb{T}}(\cdot,z)u'\right\rangle_{\mathcal{H}(\mathbb{K})^p} = \sum_{j=1}^{p}\left\langle \mathbb{K}_R(z,\zeta)M_j^R(\zeta)u,\, M_j^R(z)u'\right\rangle_{\mathcal{U}^q}. \tag{5.9}$$

Proof. Equalities (5.6), (5.7) follow immediately from (5.2), (5.3). The two other equalities follow since

$$\mathbf{Q}(z)^*\mathbb{T}(\cdot,z) = \bigoplus_{k=1}^{q}\sum_{j=1}^{p}\overline{\mathbf{q}_{jk}(z)}\mathbb{K}(\cdot,z)\begin{bmatrix} M_j^L \\ 0\end{bmatrix} = \bigoplus_{k=1}^{q}\mathbb{K}(\cdot,z)\begin{bmatrix} N_k^L(z) \\ 0\end{bmatrix} \tag{5.10}$$

by (2.10) and since

$$\mathbf{Q}(z)\widetilde{\mathbb{T}}(\cdot,z) = \bigoplus_{j=1}^{p}\sum_{k=1}^{q}\mathbf{q}_{jk}(z)\mathbb{K}(\cdot,z)\begin{bmatrix} 0 \\ N_k^R\end{bmatrix} = \bigoplus_{j=1}^{p}\mathbb{K}(\cdot,z)\begin{bmatrix} 0 \\ M_j^R(z)\end{bmatrix} \tag{5.11}$$

by (2.9). □

Definition 5.2. A contractive colligation

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D\end{bmatrix} :\ \begin{bmatrix} \mathcal{H}(\mathbb{K})^p \\ \mathcal{U}\end{bmatrix} \to \begin{bmatrix} \mathcal{H}(\mathbb{K})^q \\ \mathcal{Y}\end{bmatrix} \tag{5.12}$$

will be called *a two-component canonical functional-model* (abbreviated to **t.c.f.m.** in what follows) colligation associated with a fixed Agler decomposition (2.15) of a given $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ if

1. The state space operator A solves the structured Gleason problem

$$(\mathbf{s}f)(z) - (\mathbf{s}f)(0) = \mathbf{s}(\mathbf{Q}(z)Af)(z) = \sum_{j=1}^{p}\sum_{k=1}^{q} \mathbf{q}_{jk}(z)\,[Af]_{k,+,j}\,(z) \tag{5.13}$$

for all $f \in \mathcal{H}(\mathbb{K})^p$ whereas the adjoint operator A^* solves the dual structured Gleason problem

$$(\widetilde{\mathbf{s}}g)(z) - (\widetilde{\mathbf{s}}g)_-(0) = \widetilde{\mathbf{s}}(\mathbf{Q}(z)^*A^*g)(z) = \sum_{k=1}^{q}\sum_{j=1}^{p} \overline{\mathbf{q}_{jk}(z)}\,[A^*g]_{j,-,k}\,(z) \tag{5.14}$$

for all $g \in \mathcal{H}(\mathbb{K})^q$.

2. The operators $C : \mathcal{H}(\mathbb{K})^p \to \mathcal{Y}$, $B^* : \mathcal{H}(\mathbb{K})^q \to \mathcal{U}$ and $D : \mathcal{U} \to \mathcal{Y}$ are of the form

$$C\colon\ f \to (\mathbf{s}f)(0), \quad B^*\colon\ g \to (\widetilde{\mathbf{s}}f)(0) \quad \text{and} \quad D\colon u \to S(0)u. \tag{5.15}$$

Note that the second equalities in (5.13), (5.14) can be seen as follows:

$$\mathbf{s}(\mathbf{Q}(z)Af)(z) = \sum_{j=1}^{p} [\mathbf{Q}(z)Af]_{j,+,j}\,(z) = \sum_{j=1}^{p}\sum_{k=1}^{q} \mathbf{q}_{jk}(z)\,[Af]_{k,+,j}\,(z),$$
$$\widetilde{\mathbf{s}}(\mathbf{Q}(z)^*A^*g)(z) = \sum_{k=1}^{q} [\mathbf{Q}(z)^*A^*g]_{k,-,k}\,(z) = \sum_{k=1}^{q}\sum_{j=1}^{p} \overline{\mathbf{q}_{jk}(z)}\,[A^*g]_{j,-,k}\,(z).$$

Proposition 5.3. *Relations* (5.12), (5.14) *and* (5.15) *are equivalent respectively to equalities*

$$A^*\mathbf{Q}(\zeta)^*\mathbb{T}(\cdot,\zeta)y = \mathbb{T}(\cdot,\zeta)y - \mathbb{T}(\cdot,0)y, \tag{5.16}$$
$$A\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u = \widetilde{\mathbb{T}}(\cdot,\zeta)u - \widetilde{\mathbb{T}}(\cdot,0)u, \tag{5.17}$$
$$C^*y = \mathbb{T}(\cdot,0)y, \quad Bu = \widetilde{\mathbb{T}}(\cdot,0)u, \quad \textit{and} \quad D^*y = S(0)^*y \tag{5.18}$$

holding for every $\zeta \in \mathcal{D}_{\mathbf{Q}}$, $y \in \mathcal{Y}$ *and* $u \in \mathcal{U}$.

Proof. It follows from the first equality in (5.5) that

$$\langle(\mathbf{s}f)(z) - (\mathbf{s}f)(0),\, y\rangle_{\mathcal{Y}} = \langle f,\, \mathbb{T}(\cdot,\zeta)y - \mathbb{T}(\cdot,0)y\rangle_{\mathcal{H}(\mathbb{K})^p}$$

and on the other hand,

$$\langle\mathbf{s}(\mathbf{Q}(z)Af)(z),\, y\rangle_{\mathcal{H}(\mathbb{K})^p} = \langle\mathbf{Q}(z)Af,\, \mathbb{T}(\cdot,z)y\rangle_{\mathcal{H}(\mathbb{K})^p} = \langle f,\, A^*\mathbf{Q}(z)^*\mathbb{T}(\cdot,z)y\rangle_{\mathcal{H}(\mathbb{K})^p}\,.$$

Since the two latter equalities hold for every $f \in \mathcal{H}(\mathbb{K})^p$ and $y \in \mathcal{Y}$, the equivalence (5.13) $\Leftrightarrow$ (5.16) follows. The equivalence (5.14)$\Leftrightarrow$ (5.17) follows from (5.5) in much the same way; the formula for C^* in (5.17) follows from

$$\langle f, C^*y\rangle = \langle Cf, y\rangle = \langle (\mathbf{s}f)(0), y\rangle = \langle f, \mathbb{T}(\cdot, 0)y\rangle$$

and the formula for B is a consequence of a similar computation. The formula for D^* is self-evident. □

Proposition 5.4. *Let B, C and D be the operators defined in* (5.15). *Then*

$$CC^* + DD^* = I_{\mathcal{Y}} \quad \text{and} \quad B^*B + D^*D = I_{\mathcal{Y}}. \tag{5.19}$$

Furthermore,

$$B^* \colon \mathbf{Q}(\zeta)^*\mathbb{T}(\cdot, \zeta)y \to S(\zeta)^*y - S(0)^*y, \tag{5.20}$$

$$B^* \colon \widetilde{\mathbb{T}}(\cdot, \zeta)u \to u - S(0)^*S(\zeta)u, \tag{5.21}$$

for all $\zeta \in \mathcal{D}_{\mathbf{Q}}$, $y \in \mathcal{Y}$ and $u \in \mathcal{U}$, where $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ are defined in (5.2), (5.3).

Proof. We first observe that

$$\|C^*y\|^2 = \|\mathbb{T}(\cdot, 0)y\|^2 = \left\langle \sum_{j=1}^{p} M_j^{L*}\mathbb{K}_L(0, 0)M_j^L y, y \right\rangle = \langle (I - S(0)S(0)^*)y, y\rangle,$$

$$\|Bu\|^2 = \left\|\widetilde{\mathbb{T}}(\cdot, 0)u\right\|^2 = \left\langle \sum_{k=1}^{q} N_k^{R*}\mathbb{K}_R(0, 0)N_k^R u, u \right\rangle = \langle (I - S(0)^*S(0))u, u\rangle,$$

where the first equalities follow from formulas (5.18) for B and C^*, the second equalities follow upon letting $z = \zeta$, $u' = u$ and $y' = y$ in (5.6), (5.7), and finally, the third equalities follow from the decomposition formulas (2.11) and (2.12) evaluated at $z = \zeta = 0$. Taking into account the formulas (5.15) and (5.18) for D and D^*, we then have equalities

$$\begin{aligned} \|C^*y\|^2 &= \|y\|^2 - \|S(0)^*y\|^2 = \|y\|^2 - \|D^*y\|^2, \\ \|Bu\|^2 &= \|u\|^2 - \|S(0)u\|^2 = \|u\|^2 - \|Du\|^2 \end{aligned} \tag{5.22}$$

holding for all $y \in \mathcal{Y}$ and $u \in \mathcal{U}$ which are equivalent to operator equalities (5.19).

To verify (5.20) and (5.21) we proceed as follows. By definitions (1.11), (5.3) and (5.4) of $\mathbb{K}$, $\widetilde{\mathbb{T}}$ and $\widetilde{\mathbf{s}}$,

$$\widetilde{\mathbf{s}}\left(\widetilde{\mathbb{T}}(\cdot, \zeta)u\right) = \sum_{k=1}^{q} \left[\mathbb{K}(\cdot, \zeta)\begin{bmatrix} 0 \\ N_k^R u\end{bmatrix}\right]_{-,k} = \sum_{k=1}^{q} N_k^{R*}\mathbb{K}^R(\cdot, \zeta)N_k^R u.$$

By (5.11),

$$\widetilde{\mathbf{s}}\left(\mathbf{Q}(\zeta)^*\mathbb{T}(\cdot, \zeta)y\right) = \sum_{k=1}^{q} \left[\mathbb{K}(\cdot, \zeta)\begin{bmatrix} N_k^L(\zeta)y \\ 0\end{bmatrix}\right]_{-,k} = \sum_{k=1}^{q} N_k^{R*}\mathbb{K}^{RL}(\cdot, \zeta)N_k^L(\zeta)y.$$

Combining the definition (5.15) of B^* with the two last formulas evaluated at zero gives

$$B^*\widetilde{\mathbb{T}}(\cdot,\zeta)u = \widetilde{\mathbf{s}}\left(\widetilde{\mathbb{T}}(\cdot,\zeta)u\right)(0) = \sum_{k=1}^{q} N_k^{R*}\mathbb{K}_R(0,\zeta)N_k^R u, \tag{5.23}$$

$$B^*\mathbf{Q}(\zeta)^*\mathbb{T}(\cdot,\zeta)y = \widetilde{\mathbf{s}}\left(\mathbf{Q}(\zeta)^*\mathbb{T}(\cdot,\zeta)y\right)(0) = \sum_{k=1}^{q} N_k^{R*}\mathbb{K}_{RL}(0,\zeta)N_k^L(\zeta)y. \tag{5.24}$$

Upon letting $z=0$ in (2.14) and (2.12) and taking into account that $M_j^R(0)=0$ and $N_k^L(0)=0$, we get

$$S(\zeta)^* - S(0)^* = \sum_{k=1}^{q} N_k^{R*}\mathbb{K}_{RL}(0,\zeta)N_k^L(\zeta),$$

$$I_{\mathcal{U}} - S(0)^*S(\zeta) = \sum_{k=1}^{q} N_k^{R*}\mathbb{K}_R(0,\zeta)N_k^R \tag{5.25}$$

and combining the two latter equalities with (5.23) and (5.24) gives (5.20) and (5.21). □

Formulas (5.20), (5.21) describing the action of the operator B^* on elementary kernels of $\mathcal{D}$ were easily obtained from the general formula (5.15) for B^*. Although the operator A^* is not defined in Definition 5.2 on the whole space $\mathcal{H}(\mathbb{K})^q$, it turns out that its action on elementary kernels of $\mathcal{D}$ is completely determined by conditions (5.13) and (5.14). One half of the job is handled by formula (5.16) (which is equivalent to (5.13)). The other half is covered in the next proposition.

Proposition 5.5. *Let* $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ *be a* **t.c.f.m.** *colligation associated with the Agler decomposition* (2.15) *of a given* $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U},\,\mathcal{Y})$ *and let* $\mathbb{T}$ *be given by* (5.2)*. Then*

$$A^*\widetilde{\mathbb{T}}(\cdot,\zeta)u = \mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u - \mathbb{T}(\cdot,0)S(\zeta)u \tag{5.26}$$

for all $\zeta \in \mathcal{D}_{\mathbf{Q}}$*,* $y \in \mathcal{Y}$ *and* $u \in \mathcal{U}$*.*

Proof. We have to show that formula (5.26) follows from conditions in Definition 5.2. To this end, we first verify the equality

$$\left\|\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2_{\mathcal{H}(\mathbb{K})^p} - \left\|A\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2_{\mathcal{H}(\mathbb{K})^q} = \left\|C\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2_{\mathcal{U}}. \tag{5.27}$$

Upon letting $\zeta = z$ and $u' = u$ in (5.9) we have

$$\left\|\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2_{\mathcal{H}(\mathbb{K})^p} = \sum_{j=1}^{p}\left\langle M_j^R(\zeta)^*\mathbb{K}_R(\zeta,\zeta)M_j^R(\zeta)u,\, u\right\rangle_{\mathcal{U}^q}. \tag{5.28}$$

Making use of (5.17) (which holds by Proposition 5.3) and of (5.7) we have

$$\begin{aligned}\left\|A\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2_{\mathcal{H}(\mathbb{K})^q} &= \left\|\widetilde{\mathbb{T}}(\cdot,\zeta)u-\widetilde{\mathbb{T}}(\cdot,0)u\right\|^2_{\mathcal{H}(\mathbb{K})^q}\\ &=\sum_{k=1}^q \left\langle N_k^{R*}\left(\mathbb{K}_R(\zeta,\zeta)-\mathbb{K}_R(\zeta,0)-\mathbb{K}_R(0,\zeta)+\mathbb{K}_R(0,0)\right)N_k^R u,\, u\right\rangle_{\mathcal{U}}. \end{aligned} \tag{5.29}$$

Upon letting $z=\zeta$ in (2.12) we get the identity

$$I_{\mathcal{U}}-S(\zeta)^*S(\zeta)=\sum_{k=1}^q N_k^{R*}\mathbb{K}_R(\zeta,\zeta)N_k^R-\sum_{j=1}^p M_j^R(\zeta)^*\mathbb{K}_R(\zeta,\zeta)M_j^R(\zeta) \tag{5.30}$$

which together with equality (5.25) implies

$$\begin{aligned}&\sum_{k=1}^q N_k^{R*}\left(\mathbb{K}_R(\zeta,\zeta)-\mathbb{K}_R(\zeta,0)-\mathbb{K}_R(0,\zeta)+\mathbb{K}_R(0,0)\right)N_k^R\\ &\qquad -\sum_{j=1}^p M_j^R(\zeta)^*\mathbb{K}_R(\zeta,\zeta)M_j^R(\zeta))\\ &= I_{\mathcal{U}}-S(\zeta)^*S(\zeta)-(I_{\mathcal{U}}-S(\zeta)^*S(0))-(I_{\mathcal{U}}-S(0)^*S(\zeta))+I_{\mathcal{U}}-S(0)^*S(0)\\ &=-(S(\zeta)^*-S(0)^*)(S(\zeta)-S(0)).\end{aligned}$$

Subtracting (5.29) from (5.28) and making use of the last identity gives us

$$\left\|\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2-\left\|A\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2=\|S(\zeta)u-S(0)u\|^2_{\mathcal{Y}}. \tag{5.31}$$

On the other hand, it follows from the identity

$$S(\zeta)-S(0)=\sum_{j=1}^p M_j^{L*}\mathbb{K}_{LR}(0,\zeta)M_j^R(\zeta)$$

(which is a consequence of (2.13)), formula (5.11) and the explicit formula (5.15) for C that

$$\begin{aligned}C\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u &= \mathbf{s}\left(\bigoplus_{j=1}^p \mathbb{K}(\cdot,\zeta)\begin{bmatrix}0\\ M_j^R(\zeta)\end{bmatrix}\right)(0)\\ &=\sum_{j=1}^p M_j^{L*}\mathbb{K}_{LR}(0,\zeta)M_j^R(\zeta)=S(\zeta)u-S(0)u.\end{aligned} \tag{5.32}$$

Substituting the latter equality into (5.31) completes the proof of (5.27).

Writing (5.27) in the form

$$\left\langle (I-A^*A-C^*C)\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u,\, \mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\rangle_{\mathcal{H}(\mathbb{K})^p}=0$$

and observing that the operator $I - A^*A - C^*C$ is positive semidefinite (since $\mathbf{U}$ is contractive by Definition 5.2), we conclude that

$$(I - A^*A - C^*C)\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u \equiv 0 \quad \text{for all} \quad \zeta \in \mathcal{D}_{\mathbf{Q}},\ u \in \mathcal{U}. \tag{5.33}$$

Applying the operator C^* to both parts of (5.32) we get

$$C^*C\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u = \mathbb{T}(\cdot,0)\left(S(\zeta) - S(0)\right)u \tag{5.34}$$

by the explicit formula (5.18) for C^*. From the same formula and the formula (5.15) for D we get

$$C^*Du = C^*S(0)^*u = \mathbb{T}(\cdot,0)S(0)u. \tag{5.35}$$

We next apply the operator A^* to both parts of equality (5.17) to get

$$A^*A\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u = A^*\widetilde{\mathbb{T}}(\cdot,\zeta)u - A^*\widetilde{\mathbb{T}}(\cdot,0)u.$$

Due to the second formula in (5.18) (which holds by Proposition 5.3) the latter equality can be written as

$$A^*\widetilde{\mathbb{T}}(\cdot,\zeta)u = A^*A\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u + A^*Bu. \tag{5.36}$$

Since $\mathbf{U}$ is contractive (by Definition 5.2) and since B and D satisfy the second equality in (5.19), it then follows that $A^*B + C^*D = 0$. Thus,

$$A^*Bu = -C^*Du = -C^*S(0)^*u = -\mathbb{T}(\cdot,0)S(0)u.$$

Taking the latter equality into account and making subsequent use of (5.33)–(5.35) we then get from (5.36)

$$\begin{aligned} A^*\widetilde{\mathbb{T}}(\cdot,\zeta)u &= (I - C^*C)\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u - C^*Du \\ &= \mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u - \mathbb{T}(\cdot,0)\left(S(\zeta) - S(0)\right)u - \mathbb{T}(\cdot,0)S(0)u \\ &= \mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u - \mathbb{T}(\cdot,0)S(\zeta)u \end{aligned}$$

which completes the proof of (5.26). □

Remark 5.6. Since any **t.c.f.m.** colligation is contractive, we have in particular that $AA^* + BB^* \le I$. Therefore, formulas (5.20), (5.21) and (5.26), (5.16) defining the action of operators B^* and A^* on elementary kernels of the space $\mathcal{D}$ (see (5.46)) can be extended by continuity to define these operators on the whole space $\mathcal{D}$.

Proposition 5.7. *Any* **t.c.f.m.** *colligation* $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *associated with a fixed Agler decomposition* (2.15) *of a given* $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U},\mathcal{Y})$ *is weakly unitary and closely connected. Furthermore,*

$$S(z) = D + C(I - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)B. \tag{5.37}$$

Proof. Let $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a **t.c.f.m.** colligation of S associated with a fixed Agler decomposition (2.15). Then equalities (5.16)–(5.18) hold (by Proposition 5.3) and can be solved for $\mathbb{T}(\cdot, z)y$ and $\widetilde{\mathbb{T}}(\cdot, z)u$ as follows:

$$\mathbb{T}(\cdot, z)y = (I - A^*\mathbf{Q}(z)^*)^{-1}\mathbb{T}(\cdot, 0)y = (I - A^*\mathbf{Q}(z)^*)^{-1}C^*y, \tag{5.38}$$

$$\widetilde{\mathbb{T}}(\cdot, z)u = (I - A\mathbf{Q}(z))^{-1}\widetilde{\mathbb{T}}(\cdot, 0)u = (I - A\mathbf{Q}(z))^{-1}Bu. \tag{5.39}$$

From (5.38) and (5.20) we conclude that equalities

$$\begin{aligned}(D^* + B^*\mathbf{Q}(z)^*(I - A^*\mathbf{Q}(z)^*)^{-1}C^*)y &= S(0)^*y + B^*\mathbf{Q}(z)^*\mathbb{T}(\cdot, z)y \\ &= S(0)^*y + S(z)^*y - S(0)^*y \\ &= S(z)^*y \end{aligned} \tag{5.40}$$

hold for every $z \in \mathcal{D}_{\mathbf{Q}}$ and $y \in \mathcal{Y}$, which proves representation (5.37). Furthermore, in view of (5.2) and (5.3),

$$\begin{aligned}\mathcal{H}^{\mathcal{O}}_{C,A} &:= \bigvee \left\{\mathcal{I}^*_{p,j}(I - A^*\mathbf{Q}(z)^*)^{-1}C^*y : \ z \in \mathcal{D}_{\mathbf{Q}}, \ y \in \mathcal{Y}, \ j = 1, \dots, p\right\} \\ &= \bigvee \left\{\mathcal{I}^*_{p,j}\mathbb{T}(\cdot, z)y : \ z \in \mathcal{D}_{\mathbf{Q}}, \ y \in \mathcal{Y}, \ j = 1, \dots, p\right\} \\ &= \bigvee \left\{\mathbb{K}(\cdot, z)\begin{bmatrix}\mathbf{e}_j \otimes y \\ 0\end{bmatrix} : \ z \in \mathcal{D}_{\mathbf{Q}}, \ y \in \mathcal{Y}, \ j = 1, \dots, p\right\} \\ &= \bigvee \left\{\mathbb{K}(\cdot, z)\begin{bmatrix}\mathbf{y} \\ 0\end{bmatrix} : \ z \in \mathcal{D}_{\mathbf{Q}}, \ \mathbf{y} \in \mathcal{Y}\right\}, \\ \mathcal{H}^{\mathcal{C}}_{A,B} &:= \bigvee \left\{\mathcal{I}^*_{q,k}(I - A\mathbf{Q}(z))^{-1}Bu : \ z \in \mathcal{D}_{\mathbf{Q}}, \ u \in \mathcal{U}, \ k = 1, \dots, q\right\} \\ &= \bigvee \left\{\mathcal{I}^*_{q,k}\widetilde{\mathbb{T}}(\cdot, z)u : \ z \in \mathcal{D}_{\mathbf{Q}}, \ u \in \mathcal{U}, \ k = 1, \dots, q\right\} \\ &= \bigvee \left\{\mathbb{K}(\cdot, z)\begin{bmatrix}0 \\ \widetilde{\mathbf{e}}_k \otimes u\end{bmatrix} : \ z \in \mathcal{D}_{\mathbf{Q}}, \ u \in \mathcal{Y}, \ k = 1, \dots, q\right\} \\ &= \bigvee \left\{\mathbb{K}(\cdot, z)\begin{bmatrix}0 \\ \mathbf{u}\end{bmatrix} : \ z \in \mathcal{D}_{\mathbf{Q}}, \ \mathbf{u} \in \mathcal{U}^q\right\},\end{aligned}$$

and therefore,

$$\begin{aligned}\mathcal{H}^{\mathcal{O}}_{C,A} \bigvee \mathcal{H}^{\mathcal{C}}_{A,B} &= \bigvee \left\{\mathbb{K}(\cdot, z)\begin{bmatrix}\mathbf{y} \\ 0\end{bmatrix}, \ \mathbb{K}(\cdot, z)\begin{bmatrix}0 \\ \mathbf{u}\end{bmatrix} : \ z \in \mathcal{D}_{\mathbf{Q}}, \ \mathbf{y} \in \mathcal{Y}^p, \ \mathbf{u} \in \mathcal{U}^q\right\} \\ &= \bigvee \left\{\mathbb{K}(\cdot, z)\begin{bmatrix}\mathbf{y} \\ \mathbf{u}\end{bmatrix} : \ z \in \mathcal{D}_{\mathbf{Q}}, \ \begin{bmatrix}\mathbf{y} \\ \mathbf{u}\end{bmatrix} \in \mathcal{Y}^p \oplus \mathcal{U}^q\right\} = \mathcal{H}(\mathbb{K})\end{aligned}$$

where the last equality follows by the very construction of the reproducing kernel Hilbert space. The colligation $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is closely connected by Definition 1.7. To show that $\mathbf{U}$ is weakly unitary, let us write the Agler decomposition (2.15) for

S in the inner product form as the identity

$$\begin{aligned}
&\langle y+S(\zeta)u,\ y'+S(z)u'\rangle_{\mathcal{Y}}-\langle S(\zeta)^*y+u,\ S(z)^*y'+u'\rangle_{\mathcal{U}}\\
&=\left\langle \bigoplus_{j=1}^{p}\mathbb{K}(\cdot,\zeta)M_j(\zeta)\begin{bmatrix} y\\ u\end{bmatrix},\ \bigoplus_{j=1}^{p}\mathbb{K}(\cdot,z)M_j(z)\begin{bmatrix} y'\\ u'\end{bmatrix}\right\rangle_{\mathcal{H}(\mathbb{K})^p}\\
&\quad-\left\langle \bigoplus_{k=1}^{q}\mathbb{K}(\cdot,\zeta)N_k(\zeta)\begin{bmatrix} y\\ u\end{bmatrix},\ \bigoplus_{k=1}^{q}\mathbb{K}(\cdot,z)N_k(z)\begin{bmatrix} y'\\ u'\end{bmatrix}\right\rangle_{\mathcal{H}(\mathbb{K})^q} \qquad (5.41)
\end{aligned}$$

holding for all $z,\zeta\in\mathcal{D}_{\mathbf{Q}}$, $y,y'\in\mathcal{Y}$ and $u,u'\in\mathcal{U}$. We next observe that

$$\begin{aligned}
\bigoplus_{j=1}^{p}\mathbb{K}(\cdot,\zeta)M_j(\zeta)\begin{bmatrix} y\\ u\end{bmatrix}&=\bigoplus_{j=1}^{p}\mathbb{K}(\cdot,\zeta)\begin{bmatrix} M_j^L y\\ 0\end{bmatrix}+\bigoplus_{j=1}^{p}\mathbb{K}(\cdot,\zeta)\begin{bmatrix} 0\\ M_j^R(\zeta)u\end{bmatrix}\\
&=\mathbb{T}(\cdot,\zeta)y+\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u, \qquad (5.42)
\end{aligned}$$

where the first equality follows from definition (1.7) of $M_j(\zeta)$ and the second is a consequence of (5.2) and (5.11). Similarly, one can check the equality

$$\bigoplus_{k=1}^{q}\mathbb{K}(\cdot,\zeta)N_k(\zeta)\begin{bmatrix} y\\ u\end{bmatrix}=\mathbf{Q}(\zeta)^*\mathbb{T}(\cdot,\zeta)y+\widetilde{\mathbb{T}}(\cdot,\zeta)u,$$

which, upon being substituted together with (5.42) into (5.41), leads us to

$$\begin{aligned}
&\left\langle \begin{bmatrix} \mathbf{Q}(\zeta)^*\mathbb{T}(\cdot,\zeta)y+\widetilde{\mathbb{T}}(\cdot,\zeta)u\\ y+S(\zeta)u\end{bmatrix},\ \begin{bmatrix} \mathbf{Q}(z)^*\mathbb{T}(\cdot,z)y'+\widetilde{\mathbb{T}}(\cdot,z)u'\\ y'+S(z)u'\end{bmatrix}\right\rangle_{\mathcal{H}(\mathbb{K})^q\oplus\mathcal{Y}}\\
&=\left\langle \begin{bmatrix} \mathbb{T}(\cdot,\zeta)y+\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u\\ S(\zeta)^*y+u\end{bmatrix},\ \begin{bmatrix} \mathbb{T}(\cdot,z)y'+\mathbf{Q}(z)\widetilde{\mathbb{T}}(\cdot,z)u'\\ S(z)^*y'+u'\end{bmatrix}\right\rangle_{\mathcal{H}(\mathbb{K})^p\oplus\mathcal{U}}. \qquad (5.43)
\end{aligned}$$

Letting $u=u'=0$ and $y=y'$ in the latter equality gives

$$\left\|\begin{bmatrix} \mathbf{Q}(\zeta)^*\mathbb{T}(\cdot,\zeta)y\\ y\end{bmatrix}\right\|=\left\|\begin{bmatrix} \mathbb{T}(\cdot,\zeta)y\\ S(\zeta)^*y\end{bmatrix}\right\|$$

which on account of (5.38) can be written as

$$\left\|\begin{bmatrix} \mathbf{Q}(\zeta)^*(I-A^*\mathbf{Q}(\zeta)^*)^{-1}C^*y\\ y\end{bmatrix}\right\|=\left\|\begin{bmatrix} (I-A^*\mathbf{Q}(\zeta)^*)^{-1}C^*y\\ S(\zeta)^*y\end{bmatrix}\right\|. \qquad (5.44)$$

Since

$$\begin{bmatrix} A^* & C^*\\ B^* & D^*\end{bmatrix}\begin{bmatrix} \mathbf{Q}(\zeta)^*(I-A^*\mathbf{Q}(\zeta)^*)^{-1}C^*y\\ y\end{bmatrix}=\begin{bmatrix} (I-A^*\mathbf{Q}(\zeta)^*)^{-1}C^*y\\ S(\zeta)^*y\end{bmatrix}$$

(the top components in the latter formula are equal automatically whereas the bottom components are equal due to (5.40)), equality (5.44) tells us that $\mathbf{U}$ is weakly coisometric by Definition 1.8. Similarly letting $u = u'$ and $y = y' = 0$ in (5.43) we get

$$\left\| \begin{bmatrix} \widetilde{\mathbb{T}}(\cdot,\zeta)u \\ S(\zeta)u \end{bmatrix} \right\| = \left\| \begin{bmatrix} \mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u \\ u \end{bmatrix} \right\|$$

which in view of (5.39) can be written as

$$\left\| \begin{bmatrix} (I - A\mathbf{Q}(\zeta))^{-1}Bu \\ S(\zeta)u \end{bmatrix} \right\| = \left\| \begin{bmatrix} \mathbf{Q}(\zeta)(I - A\mathbf{Q}(\zeta))^{-1}Bu \\ u \end{bmatrix} \right\|$$

and since

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{Q}(\zeta)(I - A\mathbf{Q}(\zeta))^{-1}Bu \\ u \end{bmatrix} = \begin{bmatrix} (I - A\mathbf{Q}(\zeta))^{-1}Bu \\ S(\zeta)u \end{bmatrix}$$

(again, the top components are equal automatically and the bottom components are equal due to (5.37)), the colligation $\mathbf{U}$ is weakly isometric by Definition 1.8. □

Proposition 5.7 establishes common features of **t.c.f.m.** colligations leaving the question about the existence of at least one such colligation open. As was shown in the proof of Proposition 5.7, the Agler decomposition (2.15) can be written in the inner product form (5.43) from which we conclude that the map

$$V = \begin{bmatrix} A_V & B_V \\ C_V & D_V \end{bmatrix} : \begin{bmatrix} \mathbf{Q}(\zeta)^*\mathbb{T}(\cdot,\zeta)y + \widetilde{\mathbb{T}}(\cdot,\zeta)u \\ y + S(\zeta)u \end{bmatrix} \mapsto \begin{bmatrix} \mathbb{T}(\cdot,\zeta)y + \mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u \\ S(\zeta)^*y + u \end{bmatrix}, \tag{5.45}$$

defined completely in terms of a given Agler decomposition $\mathbb{K}$ of S, extends by linearity and continuity to an isometry from

$$\mathcal{D}_V = \bigvee \left\{ \begin{bmatrix} \mathbf{Q}(\zeta)^*\mathbb{T}(\cdot,\zeta)y \\ y \end{bmatrix}, \begin{bmatrix} \widetilde{\mathbb{T}}(\cdot,\zeta)u \\ S(\zeta)u \end{bmatrix} : \zeta \in \mathcal{D}_{\mathbf{Q}},\, y \in \mathcal{Y},\, u \in \mathcal{U} \right\}$$

onto

$$\mathcal{R}_V = \bigvee \left\{ \begin{bmatrix} \mathbb{T}(\cdot,\zeta)y \\ S(\zeta)^*y \end{bmatrix}, \begin{bmatrix} \mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u \\ u \end{bmatrix} : \zeta \in \mathcal{D}_{\mathbf{Q}},\, y \in \mathcal{Y},\, u \in \mathcal{U} \right\}.$$

It is readily seen from (2.6) that $\mathcal{D}_V$ and $\mathcal{R}_V$ contain respectively all vectors of the form $\left[\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right]$ and therefore they are split into direct sums

$$\mathcal{D}_V = \mathcal{D} \oplus \mathcal{Y} \quad \text{and} \quad \mathcal{R}_V = \mathcal{R} \oplus \mathcal{U}$$

where the subspaces $\mathcal{D} \subset \mathcal{H}(\mathbb{K})^q$ and $\mathcal{R} \subset \mathcal{H}(\mathbb{K})^p$ are given by

$$\mathcal{D} = \bigvee \left\{ \mathbf{Q}(\zeta)^*\mathbb{T}(\cdot,\zeta)y, \;\; \widetilde{\mathbb{T}}(\cdot,\zeta)u : \zeta \in \mathcal{D}_{\mathbf{Q}},\, y \in \mathcal{Y},\, u \in \mathcal{U} \right\}, \tag{5.46}$$

$$\mathcal{R} = \bigvee \left\{ \mathbb{T}(\cdot,\zeta)y, \;\; \mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u : \zeta \in \mathcal{D}_{\mathbf{Q}},\, y \in \mathcal{Y},\, u \in \mathcal{U} \right\}. \tag{5.47}$$

It follows from the reproducing kernel formulas (5.5) that the orthogonal complements to these subspaces can be described in terms of the linear maps (5.4) as

$$\mathcal{D}^{\perp} := \mathcal{H}(\mathbb{K})^q \ominus \mathcal{D} = \{g \in \mathcal{H}(\mathbb{K})^q : \ \mathbf{s}(\mathbf{Q}g) \equiv 0 \ \text{ and } \ \widetilde{\mathbf{s}}g \equiv 0\}, \tag{5.48}$$

$$\mathcal{R}^{\perp} := \mathcal{H}(\mathbb{K})^p \ominus \mathcal{R} = \{f \in \mathcal{H}(\mathbb{K})^p : \ \mathbf{s}f \equiv 0 \ \text{ and } \ \widetilde{\mathbf{s}}(\mathbf{Q}f) \equiv 0\}. \tag{5.49}$$

For the operators $A_V : \mathcal{D} \to \mathcal{R}$, $B_V : \mathcal{U} \to \mathcal{R}$, $C_V : \mathcal{D} \to \mathcal{Y}$, $D_V : \mathcal{U} \to \mathcal{Y}$ we have from (5.45) the following relations:

$$A_V \mathbf{Q}(\zeta)^* \mathbb{T}(\cdot, \zeta) y + B_V y = \mathbb{T}(\cdot, \zeta) y, \tag{5.50}$$

$$A_V \widetilde{\mathbb{T}}(\cdot, \zeta) u + B_V S(\zeta) u = \mathbf{Q}(\zeta) \widetilde{\mathbb{T}}(\cdot, \zeta) u, \tag{5.51}$$

$$C_V \mathbf{Q}(\zeta)^* \mathbb{T}(\cdot, \zeta) y + D_V y = S(\zeta)^* y, \tag{5.52}$$

$$C_V \widetilde{\mathbb{T}}(\cdot, \zeta) u + D_V S(\zeta) u = u. \tag{5.53}$$

Equalities (5.50) and (5.51) are obtained upon equating the top components in (5.45) specialized to the respective special cases $u = 0$ and $y = 0$. Equalities (5.52) and (5.53) are obtained similarly upon equating the bottom components in (5.45). Letting $\zeta = 0$ in (5.50) and (5.52) gives

$$B_V y = \mathbb{T}(\cdot, 0) y \quad \text{and} \quad D_V y = S(0)^* y. \tag{5.54}$$

Substituting the first and the second formula in (5.54) respectively into (5.50), (5.51) and into (5.52) and (5.53) results in equalities

$$A_V : \mathbf{Q}(\zeta)^* \mathbb{T}(\cdot, \zeta) y \to \mathbb{T}(\cdot, \zeta) y - \mathbb{T}(\cdot, 0) y, \tag{5.55}$$

$$A_V : \widetilde{\mathbb{T}}(\cdot, \zeta) u \to \mathbf{Q}(\zeta) \widetilde{\mathbb{T}}(\cdot, \zeta) u - \mathbb{T}(\cdot, 0) S(\zeta) u, \tag{5.56}$$

$$C_V : \mathbf{Q}(\zeta)^* \mathbb{T}(\cdot, \zeta) y \to S(\zeta)^* y - S(0)^* y, \tag{5.57}$$

$$C_V : \widetilde{\mathbb{T}}(\cdot, \zeta) u \to u - S(0)^* S(\zeta) u \tag{5.58}$$

holding for all $\zeta \in \mathcal{D}_{\mathbf{Q}}$, $u \in \mathcal{U}$ and $y \in \mathcal{Y}$ and completely defining the operators A_V and C_V on the whole space $\mathcal{D}$.

Lemma 5.8. *Given the Agler decomposition* $\mathbb{K}$ *for a function* $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$*, let* V *be the isometric operator associated with this decomposition as in* (5.45)*. A block-operator matrix* $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *of the form* (5.12) *is a* **t.c.f.m.** *colligation associated with* $\mathbb{K}$ *if and only if*

$$\|\mathbf{U}^*\| \le 1, \quad \mathbf{U}^*|_{\mathcal{D} \oplus \mathcal{Y}} = V \quad \text{and} \quad B^*|_{\mathcal{D}^{\perp}} = 0, \tag{5.59}$$

that is, $\mathbf{U}^*$ *is a contractive extension of* V *from* $\mathcal{D} \oplus \mathcal{Y}$ *to all of* $\mathcal{H}(\mathbb{K})^q \oplus \mathcal{Y}$ *subject to condition* $B^*|_{\mathcal{D}^{\perp}} = 0$*.*

Proof. Let $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a **t.c.f.m.** colligation associated with $\mathbb{K}$. Then $\mathbf{U}$ is contractive by definition and relations (5.16)–(5.18) and (5.26) hold by Propositions 5.3 and 5.5. Comparing (5.16) and (5.26) with (5.55), (5.56) we see that $A^*|_{\mathcal{D}} = A_V$. Comparing (5.20), (5.21) with (5.57), (5.58) we conclude that $B^*|_{\mathcal{D}} = C_V$. Also, it follows from (5.18) and (5.54) that $C^* = B_V$ and $D^* = D_V$. Finally, we see from formula (5.48) that $B^* f = \widetilde{\mathbf{s}} f = 0$ for every $f \in \mathcal{D}^{\perp}$, which proves the last equality in (5.59).

Conversely, let us assume that a colligation $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ meets all the conditions in (5.59). From the second relation in (5.59) we conclude the equalities (5.54)–(5.58) hold with operators A_V, B_V, C_V and D_V replaced by A^*, C^*, B^* and D^* respectively. In other words, we conclude from (5.54) that C^* and D^* are defined exactly as in (5.18) which means (by Proposition 5.3) that they are already of the requisite form. Equalities (5.57), (5.58) tell us that the operator B^* satisfies formulas (5.20), (5.21). As we have seen in the proof of Proposition 5.5, these formulas agree with the second formula in (5.15) defining B^* on the whole $\mathcal{H}(\mathbb{K})^q$. From the third condition in (5.59) we now conclude that B^* is defined by formula (5.15) on the whole $\mathcal{H}(\mathbb{K})^q$, and therefore B is also of the requisite form. The formula (5.55) (with A^* instead of A_V) leads us to (5.16) which means that A solves the Gleason problem (5.13). Then the hypotheses of Proposition 5.4 are satisfied and we conclude that the identities (5.19), (5.20) and (5.21) all hold.

To complete the proof, it remains to show that A^* solves the dual Gleason problem (5.14) or equivalently, that (5.17) holds. Rather than (5.17), what we know is equality (5.51) (with A^* and C^* instead of A_V and B_V respectively):

$$A^*\widetilde{\mathbb{T}}(\cdot,\zeta)u = \mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u - C^*S(\zeta)u \tag{5.60}$$

We use (5.60) to show that equality

$$\left\|\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2_{\mathcal{H}(\mathbb{K})^p} - \left\|A^*\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2_{\mathcal{H}(\mathbb{K})^p} = \left\|B^*\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2_{\mathcal{U}} \tag{5.61}$$

holds for every $\zeta \in \mathcal{D}_{\mathbf{Q}}$ and $u \in \mathcal{U}$. Indeed,

$$\begin{aligned}\left\|\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2 - \left\|A^*\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2 &= \left\|\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2 - \left\|\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u - C^*S(\zeta)u\right\|^2 \\ &= \left\|\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2 - \left\|\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2 - \|C^*S(\zeta)u\|^2 \\ &\quad - \left\langle C\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u,\, S(\zeta)u\right\rangle \\ &\quad - \left\langle S(\zeta)u,\, C\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\rangle.\end{aligned} \tag{5.62}$$

We next express all the terms on the right of (5.62) in terms of the function S:

$$\left\|\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2 - \left\|\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2 = \langle (I_{\mathcal{U}} - S(\zeta)^*S(\zeta))u,\, u\rangle, \tag{5.63}$$

$$\left\langle C\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u,\, S(\zeta)u\right\rangle = \langle S(\zeta)^*(S(\zeta) - S(0))u,\, u\rangle, \tag{5.64}$$

$$\left\langle S(\zeta)u,\, C\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\rangle = \langle (S(\zeta)^* - S(0)^*)\, S(\zeta)u,\, u\rangle, \tag{5.65}$$

$$\|C^*S(\zeta)u\|^2 = \|S(\zeta)u\|^2 - \|S(0)^*S(\zeta)u\|^2. \tag{5.66}$$

We mention that (5.64) follows from (5.7), (5.9) and (5.30); equality (5.63) is a consequence of (5.32). Taking adjoints in (5.64) gives (5.65) and equality (5.66) is obtained upon letting $y = S(\zeta)u$ in (5.22). We now substitute the four last equalities into (5.62) to get

$$\left\|\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2_{\mathcal{H}(\mathbb{K})^q} - \left\|A^*\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2_{\mathcal{H}(\mathbb{K})^p} = \langle R(\zeta)u,\, u\rangle_{\mathcal{U}} \tag{5.67}$$

where

$$\begin{aligned} R(\zeta) &= I_{\mathcal{U}} - S(\zeta)^*S(\zeta) + S(\zeta)^*\left(S(\zeta)-S(0)\right) \\ &\quad + \left(S(\zeta)^* - S(0)^*\right)S(\zeta) - S(\zeta)^*S(\zeta) + S(\zeta)^*S(0)S(0)^*S(\zeta) \\ &= I_{\mathcal{U}} - S(\zeta)^*S(0) - S(0)^*S(\zeta) + S(\zeta)^*S(0)S(0)^*S(\zeta) \\ &= \left(I_{\mathcal{U}} - S(\zeta)^*S(0)\right)\left(I_{\mathcal{U}} - S(0)^*S(\zeta)\right). \end{aligned}$$

By (5.21) we have

$$B^*\widetilde{\mathbb{T}}(\cdot,\zeta)u = u - S(0)^*S(\zeta)u \tag{5.68}$$

and therefore

$$\left\|B^*\widetilde{\mathbb{T}}(\cdot,\zeta)u\right\|^2_{\mathcal{U}} = \|u - S(0)^*S(\zeta)u\|^2_{\mathcal{U}} = \langle R(\zeta)u,\, u\rangle_{\mathcal{U}},$$

which together with (5.67) completes the proof of (5.61). Writing (5.61) as

$$\langle (I - AA^* - BB^*)\widetilde{\mathbb{T}}(\cdot,\zeta)u,\, \widetilde{\mathbb{T}}(\cdot,\zeta)u\rangle = 0$$

and observing that the operator $I - AA^* - BB^*$ is positive semidefinite (since $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a contraction), we conclude that

$$(I - AA^* - BB^*)\widetilde{\mathbb{T}}(\cdot,\zeta)u = 0 \quad \text{for all} \quad \zeta\in\mathcal{D}_{\mathbf{Q}},\, u\in\mathcal{U}. \tag{5.69}$$

Since the operators C and D satisfy the first equality (5.19) and since $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a contraction, we have $AC^* + BD^* = 0$. We now combine this latter equality with (5.68) and formula (5.18) for D^* to get

$$\begin{aligned} \widetilde{\mathbb{T}}(\cdot,0)u = Bu &= B(B^*\widetilde{\mathbb{T}}(\cdot,\zeta)u + S(0)^*S(\zeta)u) \\ &= BB^*\widetilde{\mathbb{T}}(\cdot,\zeta)u + BD^*S(\zeta)u \\ &= BB^*\widetilde{\mathbb{T}}(\cdot,\zeta)u - AC^*S(\zeta)u. \end{aligned} \tag{5.70}$$

We now apply the operator A to both parts of (5.60):

$$AA^*\widetilde{\mathbb{T}}(\cdot,\zeta)u = A\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u - AC^*S(\zeta)u$$

and solve the obtained identity for $A\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u$ with further simplifications based on (5.69) and (5.70):

$$\begin{aligned} A\mathbf{Q}(\zeta)\widetilde{\mathbb{T}}(\cdot,\zeta)u &= AA^*\widetilde{\mathbb{T}}(\cdot,\zeta)u + AC^*S(\zeta)u \\ &= \widetilde{\mathbb{T}}(\cdot,\zeta)u - BB^*\widetilde{\mathbb{T}}(\cdot,\zeta)u - BD^*S(\zeta)u \\ &= \widetilde{\mathbb{T}}(\cdot,\zeta)u - \widetilde{\mathbb{T}}(\cdot,0)u. \end{aligned}$$

This completes the proof of (5.17). □

As a consequence of Lemma 5.8 we get a description of all **t.c.f.m.** colligations associated with a given Agler decomposition of a Schur-Agler function.

Lemma 5.9. *Let $\mathbb{K}$ be a fixed Agler decomposition of a function $S \in \mathcal{SA}_{\mathbf{Q}}(\mathcal{U},\mathcal{Y})$. Let V be the associated isometry defined in* (5.45) *with the defect spaces $\mathcal{D}^\perp$ and $\mathcal{R}^\perp$ defined in* (5.48), (5.49)*. Then all* **t.c.f.m.** *colligations associated with $\mathbb{K}$ are of the form*

$$\mathbf{U}^* = \begin{bmatrix} X & 0 \\ 0 & V \end{bmatrix} : \begin{bmatrix} \mathcal{D}^\perp \\ \mathcal{D}\oplus\mathcal{Y} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}^\perp \\ \mathcal{R}\oplus\mathcal{U} \end{bmatrix} \tag{5.71}$$

where we have identified $\begin{bmatrix} \mathcal{H}(\mathbb{K})^q \\ \mathcal{Y} \end{bmatrix}$ *with* $\begin{bmatrix} \mathcal{D}^\perp \\ \mathcal{D}\oplus\mathcal{Y} \end{bmatrix}$ *and* $\begin{bmatrix} \mathcal{H}(\mathbb{K})^p \\ \mathcal{U} \end{bmatrix}$ *with* $\begin{bmatrix} \mathcal{R}^\perp \\ \mathcal{R}\oplus\mathcal{U} \end{bmatrix}$ *and where X is an arbitrary contraction from $\mathcal{D}^\perp$ into $\mathcal{R}^\perp$. The colligation $\mathbf{U}$ is isometric* (*coisometric, unitary*) *if and only if X is coisometric* (*isometric, unitary*)*.*

For the proof, it is enough to recall that V is unitary as an operator from $\mathcal{D}_V = \mathcal{D}\oplus\mathcal{Y}$ onto $\mathcal{R}_V = \mathcal{R}\oplus\mathcal{U}$ and then to refer to Lemma 5.8. The meaning of description (5.71) is clear: the operators B^*, C^*, D^* and the restriction of A^* to the subspace $\mathcal{D}$ in the operator colligation $\mathbf{U}^*$ are prescribed. The objective is to guarantee $\mathbf{U}^*$ be contractive by suitably defining A^* on $\mathcal{D}^\perp$. Lemma 5.9 states that $X = A^*|_{\mathcal{D}^\perp}$ must be a contraction with range contained in $\mathcal{R}^\perp$.

We now are ready to formulate the multivariable counterpart of Theorem 1.5.

Theorem 5.10. *Let S be a function in the Schur-Agler class $\mathcal{S}A_d(\mathcal{U},\mathcal{Y})$ with given Agler decomposition $\mathbb{K}$. Then*

1. *There exists a* **t.c.f.m.** *colligation $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ associated with $\mathbb{K}$.*
2. *Every* **t.c.f.m.** *colligation $\mathbf{U}$ associated with $\mathbb{K}$ is weakly unitary and closely connected and furthermore, $S(z) = D + C(I - \mathbf{Q}(z)A)^{-1}\mathbf{Q}(z)B$.*
3. *Any weakly unitary closely connected colligation $\widetilde{\mathbf{U}}$ of the form* (1.14) *with transfer function equal to S is unitarily equivalent to some* **t.c.f.m.** *colligation $\mathbf{U}$ for S.*

Proof. Part (1) is contained in Lemma 5.9. Part (2) was proved in Proposition 5.7. To prove part (3) we assume that $\widetilde{\mathbf{U}} = \left[\begin{smallmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X}^p \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X}^q \\ \mathcal{Y} \end{smallmatrix}\right]$ be a closely connected weakly unitary colligation with the state space $\mathcal{X}$ and such that

$$S(z) = D + \widetilde{C}(I - \mathbf{Q}(z)\widetilde{A})^{-1}\mathbf{Q}(z)\widetilde{B}. \tag{5.72}$$

The proof of unitary equivalence of $\widetilde{\mathbf{U}}$ to some **t.c.f.m.** colligation for S will be broken into three steps below. Let $\mathbb{G}(z)$ be the operator-valued function

$$\mathbb{G}(z) = \begin{bmatrix} \mathbb{G}_L(z) \\ \mathbb{G}_R(z) \end{bmatrix} := \begin{bmatrix} \bigoplus_{j=1}^p \widetilde{C}(I - \mathbf{Q}(z)\widetilde{A})^{-1}\mathcal{I}_{p,j} \\ \bigoplus_{k=1}^q \widetilde{B}^*(I - \mathbf{Q}(z)^*\widetilde{A}^*)^{-1}\mathcal{I}_{q,k} \end{bmatrix} \tag{5.73}$$

where the inclusion operators $\mathcal{I}_{p,j}$ and $\mathcal{I}_{q,k}$ are defined via formula (1.16). Furthermore, let $\mathbb{K}$ be the positive kernel defined by

$$\mathbb{K}(z,\zeta) = \begin{bmatrix} \mathbb{K}_L(z,\zeta) & \mathbb{K}_{LR}(z,\zeta) \\ \mathbb{K}_{RL}(z,\zeta) & \mathbb{K}_R(z,\zeta) \end{bmatrix} := \begin{bmatrix} \mathbb{G}_L(z) \\ \mathbb{G}_R(z) \end{bmatrix} \begin{bmatrix} \mathbb{G}_L(\zeta)^* & \mathbb{G}_R(\zeta)^* \end{bmatrix} \tag{5.74}$$

and let $\mathcal{H}(\mathbb{K})$ be the associated reproducing kernel Hilbert space. Let $U : \mathcal{X} \to \mathcal{H}(\mathbb{K})$ be the linear map given by

$$U : \, x \to \mathbb{G}(z)x \tag{5.75}$$

and define the operators $A \colon \mathcal{H}(\mathbb{K})^p \to \mathcal{H}(\mathbb{K})^q$, $B \colon \mathcal{U} \to \mathcal{H}(\mathbb{K})^q$ and $C \colon \mathcal{H}(\mathbb{K})^p \to \mathcal{Y}$ by

$$A\left(\oplus_{j=1}^{p} U\right) = \left(\oplus_{k=1}^{q} U\right) \widetilde{A}, \quad B = \left(\oplus_{k=1}^{q} U\right) \widetilde{B} \quad \text{and} \quad C\left(\oplus_{j=1}^{p} U\right) = \widetilde{C}. \tag{5.76}$$

Step 1: *The Agler decomposition* (2.15) *holds for the kernel* $\mathbb{K}$ *defined in* (5.74).

Step 2: *The linear map* $U \colon \mathcal{X} \to \mathcal{H}(\mathbb{K})$ *defined in* (5.75) *is unitary.*

Step 3: *The colligation* $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *with the block entries defined in* (5.76) *is a* **t.c.f.m.** *colligation associated with the Agler decomposition* $\mathbb{K}$ *for* S.

Since the colligations $\widetilde{\mathbf{U}}$ and $\mathbf{U}$ are unitarily equivalent by (5.76) and definition (1.15), part (3) of the theorem will then follow. Thus, it remains to justify the three steps.

Proof of Step 1. It follows by straightforward calculations (see, e.g., [7]) that for the transfer function S (5.72) of the colligation $\widetilde{\mathbf{U}} = \left[\begin{smallmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & D \end{smallmatrix}\right]$,

$$\begin{aligned} I - S(z)S(\zeta)^* &= \widetilde{C}(I - \mathbf{Q}(z)\widetilde{A})^{-1} \left(I - \mathbf{Q}(z)\mathbf{Q}(\zeta)^*\right) (I - \widetilde{A}^*\mathbf{Q}(\zeta)^*)^{-1}\widetilde{C}^* \\ &+ \begin{bmatrix} \widetilde{C}(I - \mathbf{Q}(z)\widetilde{A})^{-1}\mathbf{Q}(z) & I \end{bmatrix} \left(I - \widetilde{\mathbf{U}}\widetilde{\mathbf{U}}^*\right) \begin{bmatrix} \mathbf{Q}(\zeta)^*(I - \widetilde{A}^*\mathbf{Q}(\zeta)^*)^{-1}\widetilde{C}^* \\ I \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} I - S(z)^*S(\zeta) &= \widetilde{B}^*(I - \mathbf{Q}(z)^*\widetilde{A}^*)^{-1} \left(I - \mathbf{Q}(z)^*\mathbf{Q}(\zeta)\right) (I - \widetilde{A}\mathbf{Q}(\zeta))^{-1}\widetilde{B} \\ &+ \begin{bmatrix} \widetilde{B}^*(I - \mathbf{Q}(z)^*\widetilde{A}^*)^{-1}\mathbf{Q}(z)^* & I \end{bmatrix} \left(I - \widetilde{\mathbf{U}}^*\widetilde{\mathbf{U}}\right) \begin{bmatrix} \mathbf{Q}(\zeta)(I - \widetilde{A}\mathbf{Q}(\zeta))^{-1}\widetilde{B} \\ I \end{bmatrix}, \end{aligned}$$

from which it is clear that weak-coisometric property and weak-isometric properties of $\widetilde{\mathbf{U}}$ (see Definition 1.8) are exactly what is needed for the respective identities

$$I - S(z)S(\zeta)^* = \widetilde{C}(I - \mathbf{Q}(z)\widetilde{A})^{-1} \left(I - \mathbf{Q}(z)\mathbf{Q}(\zeta)^*\right) (I - \widetilde{A}^*\mathbf{Q}(\zeta)^*)^{-1}\widetilde{C}^*, \tag{5.77}$$

$$I - S(z)^*S(\zeta) = \widetilde{B}^*(I - \mathbf{Q}(z)^*\widetilde{A}^*)^{-1} \left(I - \mathbf{Q}(z)^*\mathbf{Q}(\zeta)\right) (I - \widetilde{A}\mathbf{Q}(\zeta))^{-1}\widetilde{B}. \tag{5.78}$$

Since $\widetilde{\mathbf{U}}$ is weakly unitary, the two latter identities hold. Also we observe that for S of the form (5.72),

$$\begin{aligned} S(z) - S(\zeta) &= \widetilde{C}(I - \mathbf{Q}(z)\widetilde{A})^{-1}\mathbf{Q}(z)\widetilde{B} - \widetilde{C}\mathbf{Q}(\zeta)(I - \widetilde{A}\mathbf{Q}(\zeta))^{-1}\widetilde{B} \\ &= \widetilde{C}(I - \mathbf{Q}(z)\widetilde{A})^{-1} \left(\mathbf{Q}(z) - \mathbf{Q}(\zeta)\right) (I - \widetilde{A}\mathbf{Q}(\zeta))^{-1}\widetilde{B}. \end{aligned} \tag{5.79}$$

On the other hand, for the function $\mathbb{G}_L$ defined in (5.73) and for M_j^L given in (2.8), we have

$$M_j^{L*}\mathbb{G}_L(z) = \widetilde{C}(I - \mathbf{Q}(z)\widetilde{A})^{-1}\mathcal{I}_{p,j}.$$

Then we conclude from (2.10) that

$$N_k^L(z)^*\mathbb{G}_L(z) = \sum_{j=1}^p \mathbf{q}_{jk}(z)M_j^{L*}\mathbb{G}_L(z) = \widetilde{C}(I-\mathbf{Q}(z)\widetilde{A})^{-1}\sum_{j=1}^p \mathbf{q}_{jk}(z)\mathcal{I}_{p,j}$$
$$= \widetilde{C}(I-\mathbf{Q}(z)\widetilde{A})^{-1}\mathbf{Q}(z)\mathcal{I}_{q,k}$$

and since $\sum_{j=1}^p \mathcal{I}_{p,j}\mathcal{I}_{p,j}^* = I_{\mathcal{X}^p}$ and $\sum_{k=1}^q \mathcal{I}_{q,k}\mathcal{I}_{q,k}^* = I_{\mathcal{X}^q}$, we have for the kernel $\mathbb{K}_L$ defined in (5.74),

$$\sum_{j=1}^p M_j^{L*}\mathbb{K}_L(z,\zeta)M_j^L - \sum_{k=1}^q N_k^L(z)^*\mathbb{K}_L(z,\zeta)N_k^L(\zeta)$$
$$= \sum_{j=1}^p M_j^{L*}\mathbb{G}_L(z)\mathbb{G}_L(\zeta)^*M_j^L - \sum_{k=1}^q N_k^L(z)^*\mathbb{G}_L(z)\mathbb{G}_L(\zeta)^*N_k^L(\zeta)$$
$$= \sum_{j=1}^p \widetilde{C}(I-\mathbf{Q}(z)\widetilde{A})^{-1}\mathcal{I}_{p,j}\mathcal{I}_{p,j}^*(I-\widetilde{A}^*\mathbf{Q}(\zeta)^*)^{-1}\widetilde{C}^*$$
$$- \sum_{k=1}^q \widetilde{C}(I-\mathbf{Q}(z)\widetilde{A})^{-1}\mathbf{Q}(z)\mathcal{I}_{q,k}\mathcal{I}_{q,k}^*\mathbf{Q}(\zeta)^*(I-\widetilde{A}^*\mathbf{Q}(\zeta)^*)^{-1}\widetilde{C}^*$$
$$= \widetilde{C}(I-\mathbf{Q}(z)\widetilde{A})^{-1}\left(I-\mathbf{Q}(z)\mathbf{Q}(\zeta)^*\right)(I-\widetilde{A}^*\mathbf{Q}(\zeta)^*)^{-1}\widetilde{C}^*.$$

Comparing the last equality with (5.77) we get the left Agler decomposition (2.11) for the kernel $\mathbb{K}_L$ from (5.73). It can be shown in much the same way that (5.78) and (5.79) are equivalent to (2.12) and (2.13) respectively (with $\mathbb{K}_R$ and $\mathbb{K}_{LR}$ chosen as in (5.73)). Since (2.14) follows upon taking conjugates in (2.13), we arrive at (2.15).

Proof of Step 2. Due to factorization $\mathbb{K}(z,\zeta) = \mathbb{G}(z)\mathbb{G}(\zeta)^*$ (see (5.74)), the reproducing kernel Hilbert space $\mathcal{H}(\mathbb{K})$ can be characterized as the range space $\mathcal{H}(\mathbb{K}) = \{f(z) = \mathbb{G}(z)x : \ x \in \mathcal{X}\}$ with the lifted norm $\|\mathbb{G}x\|_{\mathcal{H}(\mathbb{K})} = \|(I-\pi)x\|_{\mathcal{X}}$ where π is the orthogonal projection onto the subspace $\mathcal{X}^\circ = \{x \in \mathcal{X} : \mathbb{G}x \equiv 0\}$. For every vector $x \in \mathcal{X}^\circ$ we have by (5.73),

$$\widetilde{C}(I-\mathbf{Q}(z)\widetilde{A})^{-1}\mathcal{I}_{p,j}x = 0 \quad \text{and} \quad \widetilde{B}^*(I-\mathbf{Q}(z)^*\widetilde{A}^*)^{-1}\mathcal{I}_{q,k}x = 0$$

for all $j = 1,\dots,p$ and $k = 1,\dots,q$. Then x is orthogonal to the spaces $\mathcal{H}^{\mathcal{O}}_{\widetilde{C},\widetilde{A}}$ and $\mathcal{H}^{\mathcal{C}}_{\widetilde{A},\widetilde{B}}$ (see Definition 1.7) and since the colligation $\widetilde{\mathbf{U}}$ is closely connected, it follows that $x = 0$. Thus, $\mathcal{X}^\circ$ is trivial and $\|\mathbb{G}x\|_{\mathcal{H}(\mathbb{K})} = \|x\|_{\mathcal{X}}$ which means that the operator $U : x \to \mathbb{G}(z)x$ is a unitary operator from $\mathcal{X}$ to $\mathcal{H}(\mathbb{K})$.

Proof of Step 3. We first observe that the colligation $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ is a contraction since it is unitarily equivalent to a weakly unitary colligation $\widetilde{\mathbf{U}}$. It remains to show that A solves the Gleason problems (5.13), (5.14) and that C and B^* are of the form (5.15).

Take the generic element f of $\mathcal{H}(\mathbb{K})^p$ in the form

$$f(z) = \bigoplus_{j=1}^{p} \mathbb{G}(z)x_j \quad \text{and let} \quad \mathbf{x} := \bigoplus_{j=1}^{p} x_j \in \mathcal{X}^p, \tag{5.80}$$

so that $f = (\oplus_{j=1}^{p} U)\mathbf{x}$ by (5.75), or equivalently, $\mathbf{x} = (\oplus_{j=1}^{p} U^*)f$, since U is unitary. By definitions (5.4) and (5.73) we have

$$(\mathbf{s}f)(z) = \sum_{j=1}^{p} \widetilde{C}(I - \mathbf{Q}(z)\widetilde{A})^{-1}\mathcal{I}_{p,j}x_j = \widetilde{C}(I - \mathbf{Q}(z)\widetilde{A})^{-1}\mathbf{x}. \tag{5.81}$$

Upon evaluating the latter equality at $z = 0$ and taking into account (2.6), we get for the operator C from (5.76)

$$Cf = \widetilde{C}(\oplus_{j=1}^{p} U^*)f = \widetilde{C}\mathbf{x} = (\mathbf{s}f)(0)$$

so that the formula (5.15) for C holds. We also have from (5.81)

$$(\mathbf{s}f)(z) - (\mathbf{s}f)(0) = \widetilde{C}(I - \mathbf{Q}(z)\widetilde{A})^{-1}\mathbf{x} - \widetilde{C}\mathbf{x} = \widetilde{C}(I - \mathbf{Q}(z)\widetilde{A})^{-1}\mathbf{Q}(z)\widetilde{A}\mathbf{x}. \tag{5.82}$$

On the other hand, for the operator A defined in (5.76), we have

$$\mathbf{Q}(z)Af = \mathbf{Q}(z)A\left(\oplus_{j=1}^{p} U\right)\mathbf{x} = \mathbf{Q}(z)(\oplus_{k=1}^{q} U)\widetilde{A}\mathbf{x} = (\oplus_{k=1}^{p} U)\mathbf{Q}(z)\widetilde{A}\mathbf{x}$$

and therefore, by formula (5.81) applied to $\mathbf{Q}(z)\widetilde{A}\mathbf{x}$ rather than to $\mathbf{x}$ we get

$$\mathbf{s}(\mathbf{Q}(z)Af)(z) = \widetilde{C}(I - \mathbf{Q}(z)\widetilde{A})^{-1}\mathbf{Q}(z)\widetilde{A}\mathbf{x}$$

which together with (5.82) implies (5.13).

We now take the generic element g of $\mathcal{H}(\mathbb{K})^q$ in the form

$$g(z) = \bigoplus_{k=1}^{q} \mathbb{G}(z)\widetilde{x}_k \quad \text{where now} \quad \widetilde{\mathbf{x}} := \bigoplus_{k=1}^{q} \widetilde{x}_k \in \mathcal{X}^q, \tag{5.83}$$

so that $\widetilde{\mathbf{x}} = (\oplus_{k=1}^{q} U^*)g$. By definitions (5.4) and (5.73) we have

$$(\widetilde{\mathbf{s}}g)(z) = \sum_{k=1}^{q} \widetilde{B}^*(I - \mathbf{Q}(z)^*\widetilde{A}^*)^{-1}\widetilde{\mathcal{I}}_{q,k}\widetilde{x}_k = \widetilde{B}^*(I - \mathbf{Q}(z)^*\widetilde{A}^*)^{-1}\widetilde{\mathbf{x}}. \tag{5.84}$$

Upon evaluating the latter equality at $z = 0$ and taking into account (2.6), we get for the operator B^* from (5.76)

$$B^*g = \widetilde{B}^*(\oplus_{k=1}^{q} U^*)g = \widetilde{B}^*\widetilde{\mathbf{x}} = (\widetilde{\mathbf{s}}g)(0)$$

so that the formula (5.15) for B^* holds. We also have from (5.84)

$$(\widetilde{\mathbf{s}}g)(z) - (\widetilde{\mathbf{s}}g)(0) = \widetilde{B}^*(I - \mathbf{Q}(z)^*\widetilde{A}^*)^{-1}\mathbf{Q}(z)^*\widetilde{A}^*\widetilde{\mathbf{x}}. \tag{5.85}$$

On the other hand, for the operator A defined in (5.76), we have

$$\mathbf{Q}(z)^*A^*g = \mathbf{Q}(z)^*A^*(\oplus_{k=1}^{q} U)\widetilde{\mathbf{x}} = \mathbf{Q}(z)^*(\oplus_{j=1}^{p} U)\widetilde{A}^*\widetilde{\mathbf{x}} = (\oplus_{k=1}^{q} U)\mathbf{Q}(z)^*\widetilde{A}^*\widetilde{\mathbf{x}}$$

and therefore, by formula (5.84) applied to $\mathbf{Q}(z)^*\widetilde{A}^*\widetilde{\mathbf{x}}$ instead of $\widetilde{\mathbf{x}}$ we get

$$\widetilde{\mathbf{s}}(\mathbf{Q}(z)^*A^*g)(z) = \widetilde{B}^*(I - \mathbf{Q}(z)^*\widetilde{A}^*)^{-1}\mathbf{Q}^*(z)\widetilde{A}^*\widetilde{\mathbf{x}}$$

which together with (5.85) implies (5.14). This completes the proof of Step 3 and therefore, of the theorem. □

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Generalized Lax Pair Operator Method and Nonautonomous Solitons

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Abstract. The generalized Lax pair operator method and the concept of nonautonomous solitons in nonlinear and dispersive nonautonomous physical systems are introduced. Novel soliton solutions for the nonautonomous nonlinear Schrödinger equation (NLSE) models with linear and harmonic oscillator potentials substantially extend the concept of classical solitons and generalize it to the plethora of nonautonomous solitons that interact elastically and generally move with varying amplitudes, speeds and spectra adapted both to the external potentials and to the dispersion and nonlinearity variations. The concept of the designable integrability of the variable coefficients nonautonomous NLSE is introduced. The nonautonomous soliton concept and the designable integrability can be applied to different physical systems, from hydrodynamics and plasma physics to nonlinear optics and matter-waves and offer many opportunities for further scientific studies

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1. Introduction

Zabusky and Kruskal [1] introduced for the first time the soliton concept to characterize nonlinear solitary waves that do not disperse and preserve their identity during propagation and after a collision. The Greek ending "on" is generally used to describe elementary particles and this word was introduced to emphasize the most remarkable feature of these solitary waves. This means that the energy can propagate in the localized form and that the solitary waves emerge from the interaction completely preserved in form and speed with only a phase shift. Because of these defining features, the classical soliton is being considered as the ideal natural data bit. The optical soliton in fibers presents a beautiful example in which an abstract mathematical concept has produced a large impact on the real world of high technologies [2–6].

The classical soliton concept was developed for nonlinear and dispersive systems that have been autonomous; namely, time has only played the role of the independent variable and has not appeared explicitly in the nonlinear evolution equation. A not uncommon situation is one in which a system is subjected to some form of external time-dependent force. Such situations could include repeated stress testing of a soliton in nonuniform media with time-dependent density gradients, these situations are typical both for experiments with temporal/spatial optical solitons, soliton lasers and ultrafast soliton switches and logic gates [2–6].

How can we determine whether a given nonlinear evolution equation is integrable or not? The ingenious method to answer this question was discovered by Gardner, Green, Kruskal and Miura (GGKM) [7]. Following this work, Lax [8] formulated a general principle for associating of nonlinear evolution equations with linear operators, so that the eigenvalues of the linear operator are integrals of the nonlinear equation. Lax developed the method of inverse scattering transform (IST) based on an abstract formulation of evolution equations and certain properties of operators in a Hilbert space, some of which are well known in the context of quantum mechanics. Ablowitz, Kaup, Newell, Segur (AKNS) [9] have found that many physically meaningful nonlinear models can be solved by IST method.

In the traditional scheme of the IST method, the spectral parameter Λ of the auxiliary linear problem is assumed to be a time independent constant $\Lambda'_t = 0$, and this fact plays a fundamental role in the development of analytical theory [10]. The nonlinear evolution equations that arise in the approach of variable spectral parameter, $\Lambda'_t \neq 0$, contain, as a rule, some coefficients explicitly dependent on time. The IST method with variable spectral parameter makes it possible to construct not only the well-known models for nonlinear autonomous physical systems, but also discover many novel integrable and physically significant nonlinear nonautonomous equations.

Historically, the study of soliton propagation through density gradients began with the pioneering work of Tappert and Zabusky [11]. As early as in 1976 Chen and Liu [12] substantially extended the concept of classical solitons to the accelerated motion of a soliton in a linearly inhomogeneous plasma. It was discovered that for the nonlinear Schrödinger equation model (NLSE) with a linear external potential, the IST method can be generalized by allowing the time-varying eigenvalue (TVE), and as a consequence of this, the solitons with time-varying velocities (but with time invariant amplitudes) have been predicted [12]. At the same time Calogero and Degaspieris [13] introduced the general class of soliton solutions for the nonautonomous Korteweg-de Vries (KdV) models with varying nonlinearity and dispersion. It was shown that the basic property of solitons, to interact elastically, was also preserved, but the novel phenomenon was demonstrated, namely the fact that each soliton generally moves with variable speed as a particle acted by an external force rather than as a free particle [13]. In particular, to appreciate the significance of this analogy, Calogero and Degaspieris introduced the terms boomeron and trappon instead of classical KdV solitons [13]. Some analytical approaches for the soliton solutions of the NLSE in the nonuniform medium were developed by Gupta and Ray [14], Herrera [15], and Balakrishnan [16].

More recently, different aspects of soliton dynamics described by the nonautonomous NLSE models were investigated in [17, 18]. The "ideal" soliton-like interaction scenarios among solitons have been studied in [17, 18] within the generalized nonautonomous NLSE models with varying dispersion, nonlinearity and dissipation or gain. One important step was performed recently by Zhao, Luo and Chai [19–22] and Shin [23]. It is well known that a nonlinear partial differential equation is solvable by the IST method if every ordinary differential equation derived from it satisfies the Painlevé property. The Painlevé analysis can be considered as a practical test for the existence of the IST.

In this work, we clarify our algorithm based on the Lax pair generalization and reveal generic properties of nonautonomous solitons. We consider the generalized nonautonomous NLSE models with varying nonlinearities from the point of view of their exact integrability both for confining and expulsive external potentials. To test the validity of our predictions, the experimental arrangement should be inspected to be as close as possible to the optimal map of parameters at which the problem proves to be exactly integrable [24–26].

2. Lax operator method formulation and exact integrability of nonautonomous nonlinear and dispersive models with external potentials

The classification of dynamic systems into autonomous and nonautonomous is often convenient and can correspond to different physical situations in which, respectively, external time-dependent driving force is present or absent. The mathematical treatment of nonautonomous system of equations is considerably more complicated then the treatment of autonomous ones. As a typical illustration we may mention both a simple pendulum whose length changes with time and parametrically driven nonlinear Duffing oscillator [27].

In the framework of the IST method the nonlinear integrable equation arises as the compatibility condition of the system of the eigenvalue linear matrix differential equations

$$\psi_x = \widehat{\mathcal{F}}\psi(x,t), \qquad \psi_t = \widehat{\mathcal{G}}\psi(x,t).$$

Here $\psi(x,t) = \{\psi_1, \psi_2\}^T$ is 2-component complex function, $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{G}}$ are complex-valued (2×2) matrices.

Considering the general case of the IST method with time-dependent spectral parameter $\Lambda(T)$ and taking matrices $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{G}}$ in the form

$$\widehat{\mathcal{F}}(\Lambda)=\widehat{\mathcal{F}}\left\{\Lambda(T),q\left[S(X,T),T\right];\frac{\partial q}{\partial S}\left(\frac{\partial S}{\partial X}\right);\frac{\partial^2 q}{\partial S^2}\left(\frac{\partial S}{\partial X}\right)^2;\ldots;\frac{\partial^n q}{\partial S^n}\left(\frac{\partial S}{\partial X}\right)^n\right\}$$

$$\widehat{\mathcal{G}}(\Lambda)=\widehat{\mathcal{G}}\left\{\Lambda(T),q\left[S(X,T),T\right];\frac{\partial q}{\partial S}\left(\frac{\partial S}{\partial X}\right);\frac{\partial^2 q}{\partial S^2}\left(\frac{\partial S}{\partial X}\right)^2;\ldots;\frac{\partial^n q}{\partial S^n}\left(\frac{\partial S}{\partial X}\right)^n\right\}$$

where $S=S(x,t)$ and $T(t)=t$ are generalized (dependent) coordinates, whereas x and t are two independent variables, and the function $q\left[S(X,T),T\right]$ denotes scattering potentials $Q(S,T)$ or $R(S,T)$, let us represent the desired nonlinear evolution equation as the condition for the compatibility of the pair of linear differential equations, to which the inverse scattering method can be applied

$$\frac{\partial\widehat{\mathcal{F}}}{\partial T}+\frac{\partial\widehat{\mathcal{F}}}{\partial S}S_t-\frac{\partial\widehat{\mathcal{G}}}{\partial S}S_x+\left[\widehat{\mathcal{F}},\widehat{\mathcal{G}}\right]=0, \tag{1}$$

where

$$\widehat{\mathcal{F}}=-i\Lambda(T)\widehat{\sigma}_3+\widehat{U}\widehat{\phi}, \tag{2}$$

$$\widehat{\mathcal{G}}=\begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{3}$$

$\widehat{\sigma_3}$ is the Pauli spin matrix and matrices $\widehat{U}$ and $\widehat{\phi}$ are given by

$$\widehat{U}=\sqrt{\sigma}F^{\gamma}\left(T\right)\begin{pmatrix} 0 & Q(S,T) \\ R(S,T) & 0 \end{pmatrix}, \tag{4}$$

$$\widehat{\phi}=\begin{pmatrix} \exp[-i\varphi/2] & 0 \\ 0 & \exp[i\varphi/2] \end{pmatrix}. \tag{5}$$

Here $F(T)$ and $\varphi(S,T)$ are real unknown functions, γ is an arbitrary constant, and $\sigma=\pm1$. The desired AKNS elements of $\widehat{\mathcal{G}}$ matrix: $\widehat{\mathcal{G}}=\sum_{k=0}^{k=3}G_k\Lambda^k$,can be constructed in the form

$$A=A_0+A_1\Lambda+A_2\Lambda^2+A_3\Lambda^3$$
$$B=B_0+B_1\Lambda+B_2\Lambda^2+B_3\Lambda^3$$
$$C=C_0+C_1\Lambda+C_2\Lambda^2+C_3\Lambda^3$$

with time varying spectral parameter given by

$$\Lambda_T=\lambda_0\left(T\right)+\lambda_1\left(T\right)\Lambda\left(T\right),$$

where time-dependent functions $\lambda_0\left(T\right)$ and $\lambda_1\left(T\right)$ are the expansion coefficients of Λ_T in powers of the spectral parameter $\Lambda\left(T\right)$.

Substituting Eqs. (2–5) into Eq. (1), we obtain

$$A_S S_x=\,-i\Lambda_T+\sqrt{\sigma}F^{\gamma}\left(CQe^{+}-BRe^{-}\right), \tag{6}$$

$$\begin{aligned}\sqrt{\sigma}F^{\gamma}e^{+}Q_SS_t = & -\frac{i}{2}\sqrt{\sigma}F^{\gamma}e^{+}Q\left(\varphi_T+\varphi_SS_t\right)-\sqrt{\sigma}F^{\gamma}e^{+}Q_T\\ & -\sqrt{\sigma}F^{\gamma}e^{+}Q\left(\gamma\frac{F_T}{F}\right)+2i\Lambda B+2\sqrt{\sigma}F^{\gamma}e^{+}AQ+B_SS_x,\end{aligned}\quad (7)$$

$$\begin{aligned}\sqrt{\sigma}F^{\gamma}e^{-}R_SS_t = & +\frac{i}{2}\sqrt{\sigma}F^{\gamma}e^{-}R\left(\varphi_T+\varphi_SS_t\right)-\sqrt{\sigma}F^{\gamma}e^{-}R_T\\ & -\sqrt{\sigma}F^{\gamma}e^{-}R\left(\gamma\frac{F_T}{F}\right)-2i\Lambda C-2\sqrt{\sigma}AF^{\gamma}Re^{-}+C_SS_x,\end{aligned}\quad (8)$$

where we introduce the following notations $e^{\pm}=\exp[\pm i\varphi S/2]$.

Solving the system (6–8), we find both the matrix elements $A,\ B,\ C$

$$\begin{aligned}A = & -i\lambda_0S/S_x+a_0-\frac{1}{4}a_3\sigma F^{2\gamma}(QR\varphi_SS_x+iQR_SS_x-iRQ_SS_x)\\ & +\frac{1}{2}a_2\sigma F^{2\gamma}QR+\Lambda\left(-i\lambda_1S/S_x+\frac{1}{2}a_3\sigma F^{2\gamma}QR+a_1\right)\\ & +a_2\Lambda^2+a_3\Lambda^3,\\ B = & \sqrt{\sigma}F^{\gamma}e^{+}\Bigg\{-\frac{i}{4}a_3S_x^2\left(Q_{SS}+\frac{i}{2}Q\varphi_{SS}-\frac{1}{4}Q\varphi_S^2+iQ_S\varphi_S\right)\\ & -\frac{i}{4}a_2Q\varphi_SS_x-\frac{1}{2}a_2Q_SS_x\\ & +iQ\left(-i\lambda_1S/S_x+\frac{1}{2}a_3\sigma F^{2\gamma}QR+a_1\right)\\ & +\Lambda\left(-\frac{i}{4}a_3Q\varphi_SS_x-\frac{1}{2}a_3Q_SS_x+ia_2Q\right)+ia_3\Lambda^2Q\Bigg\},\\ C = & \sqrt{\sigma}F^{\gamma}e^{-}\Bigg\{-\frac{i}{4}a_3S_x^2\left(R_{SS}-\frac{i}{2}R\varphi_{SS}-\frac{1}{4}R\varphi_S^2-iR_S\varphi_S\right)\\ & -\frac{i}{4}a_2R\varphi_SS_x+\frac{1}{2}a_2R_SS_x\\ & +iR\left(-i\lambda_1S/S_x+\frac{1}{2}a_3\sigma F^{2\gamma}QR+a_1\right)\\ & +\Lambda\left(-\frac{i}{4}a_3R\varphi_SS_x+\frac{1}{2}a_3R_SS_x+ia_2R\right)+ia_3\Lambda^2R\Bigg\},\end{aligned}$$

and two general equations

$$\begin{aligned}iQ_T = & \frac{1}{4}a_3Q_{SSS}S_x^3+\frac{3i}{8}a_3Q_{SS}\varphi_SS_x^3\\ & -\frac{3i}{4}a_3\sigma F^{2\gamma}Q^2R\varphi_SS_x-\frac{3}{2}a_3\sigma F^{2\gamma}QRQ_SS_x\\ & -\frac{i}{2}a_2Q_{SS}S_x^2+ia_2\sigma F^{2\gamma}Q^2R\end{aligned}\quad (9)$$

$$
\begin{aligned}
&+ iQ_S\left(-S_t + \lambda_1 S + ia_1 S_x - \frac{i}{2}a_2\varphi_S S_x^2 + \frac{3}{8}a_3\varphi_{SS}S_x^3 + \frac{3i}{16}a_3\varphi_S^2 S_x^3\right)\\
&+ Q\left(i\lambda_1 - i\gamma\frac{F_T}{F} + \frac{1}{2}a_2\varphi_{SS}S_x^2 - \frac{3}{16}a_3\varphi_S\varphi_{SS}S_x^3\right)\\
&+ Q\left[2\lambda_0 S/S_x + 2ia_0 + \frac{1}{2}\left(\varphi_T + \varphi_S S_t\right) - \frac{1}{2}\lambda_1 S\varphi_S - \frac{i}{2}a_1\varphi_S S_x\right]\\
&+ Q\left(\frac{i}{8}a_2\varphi_S^2 S_x^2 - \frac{i}{32}a_3\varphi_S^3 S_x^3 + \frac{i}{8}a_3\varphi_{SSS}S_x^3\right)
\end{aligned}
$$

$$
\begin{aligned}
iR_T = &\frac{1}{4}a_3 R_{SSS}S_x^3 - \frac{3i}{8}a_3 R_{SS}\varphi_S S_x^3\\
&+ \frac{3i}{4}a_3\sigma F^{2\gamma}R^2 Q\varphi_S S_x - \frac{3}{2}a_3\sigma F^{2\gamma}R^2 Q_S S_x\\
&+ \frac{i}{2}a_2 R_{SS}S_x^2 - ia_2\sigma F^{2\gamma}R^2 Q\\
&+ iR_S\left(-S_t + \lambda_1 S + ia_1 S_x - \frac{i}{2}a_2\varphi_S S_x^2 - \frac{3}{8}a_3\varphi_{SS}S_x^3 + \frac{3i}{16}a_3\varphi_S^2 S_x^3\right)\\
&+ R\left(i\lambda_1 - i\gamma\frac{F_T}{F} + \frac{1}{2}a_2\varphi_{SS}S_x^2 - \frac{3}{16}a_3\varphi_S\varphi_{SS}S_x^3\right)\\
&+ R\left[-2\lambda_0 S/S_x - 2ia_0 - \frac{1}{2}\left(\varphi_T + \varphi_S S_t\right) + \frac{1}{2}\lambda_1 S\varphi_S + \frac{i}{2}a_1\varphi_S S_x\right]\\
&+ R\left(-\frac{i}{8}a_2\varphi_S^2 S_x^2 + \frac{i}{32}a_3\varphi_S^3 S_x^3 - \frac{i}{8}a_3\varphi_{SSS}S_x^3\right),
\end{aligned}
\tag{10}
$$

where the arbitrary time-dependent functions $a_0(T),\ a_1(T),\ a_2(T),\ a_3(T)$ have been introduced.

By using the following reduction procedure $R = -Q^*$, it is easy to find that two equations (9) and (10) take the same form if the following conditions

$$
\begin{gathered}
a_0 = -a_0^*, \quad a_1 = -a_1^*, \quad a_2 = -a_2^*, \quad a_3 = -a_3^*,\\
\lambda_0 = \lambda_0^*, \quad \lambda_1 = \lambda_1^*, \quad F = F^*
\end{gathered}
\tag{11}
$$

are fulfilled.

3. Generalized nonlinear Schrödinger equation and solitary waves in nonautonomous nonlinear and dispersive systems: nonautonomous solitons

Let us study a special case of the reduction procedure for Eqs. (9–10) where $a_3 = 0$

$$
\begin{aligned}
A = &- i\lambda_0 S/S_x + a_0(T) - \frac{1}{2}a_2(T)\sigma F^{2\gamma}\left|Q\right|^2\\
&- i\lambda_1 S/S_x\Lambda + a_1(T)\Lambda + a_2(T)\Lambda^2,
\end{aligned}
$$

$$B = \sqrt{\sigma} F^{\gamma} \exp(i\varphi/2) \left\{ -\frac{i}{4} a_2(T) Q \varphi_S S_x - \frac{1}{2} a_2(T) Q_S S_x \right\},$$
$$+ i \left\{ Q \left[-i\lambda_1 S/S_x + a_1(T) + \Lambda a_2(T) \right] \right\}$$
$$C = \sqrt{\sigma} F^{\gamma} \exp(-i\varphi/2) \left\{ \frac{i}{4} a_2(T) Q^* \varphi_S S_x - \frac{1}{2} a_2(T) Q_S^* S_x \right\}$$
$$- i \left\{ Q^* \left[-i\lambda_1 x + a_1(T) + \Lambda a_2(T) \right] \right\}.$$

In accordance with conditions (11), the imaginary functions $a_0(T),\ a_1(T),\ a_2(T)$ can be defined in the following way

$$a_0(T) = i\gamma_0(T), \quad a_1(T) = iV(T),$$
$$a_2(T) = -iD_2(T),\ R_2(T) = F^{2\gamma} D_2(T)$$

where $D_2(T), V(T),\ \gamma_0(T)$ are arbitrary real functions. The coefficients $D_2(T)$ and $N_2(T)$ are positively defined functions (for $\sigma = -1$, γ is assumed as a semi-entire number).

Thus, Eqs. (9–10) can be transformed into

$$\begin{aligned} iQ_T = & -\frac{1}{2} D_2(T) Q_{SS} S_x^2 - \sigma R_2(T) |Q|^2 Q \\ & - iQ_S \left(\frac{1}{2} D_2 \varphi_S S_x^2 + S_t + V S_x - \lambda_1 S \right) \\ & + Q \left(\frac{1}{8} D_2 \varphi_S^2 S_x^2 - 2\gamma_0(T) + 2\lambda_0 S/S_x \right) \\ & + Q \left(\frac{1}{2} \varphi_T + \frac{1}{2} \varphi_S S_t + \frac{1}{2} V \varphi_S S_x - \frac{1}{2} \lambda_1 \varphi_S S \right) \\ & + iQ \left(-\frac{1}{4} D_2 \varphi_{SS} S_x^2 - \gamma \frac{F_T}{F} + \lambda_1 \right) \end{aligned}$$

or

$$\begin{aligned} iQ_T = & -\frac{1}{2} D_2 Q_{SS} S_x^2 - \sigma R_2 |Q|^2 Q \\ & - i\widetilde{V} Q_S + i\Gamma Q + UQ, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \widetilde{V}(S,T) &= \frac{1}{2} D_2 S_x^2 \varphi_S + V S_x + S_t - \lambda_1 S, \\ U(S,T) &= \frac{1}{8} D_2 S_x^2 \varphi_S^2 - 2\gamma_0 + \frac{1}{2} \left(\varphi_T + \varphi_S S_t + V S_x \varphi_S \right) \\ & \quad + 2\lambda_0 S/S_x - \frac{1}{2} \lambda_1 \varphi_S S, \end{aligned} \tag{13}$$

$$\begin{aligned} \Gamma &= \left(-\gamma \frac{F_T}{F} - \frac{1}{4} D_2 S_x^2 \varphi_{SS} + \lambda_1 \right) \\ &= \left(\frac{1}{2} \frac{W(R_2, D_2)}{R_2 D_2} - \frac{1}{4} D_2 S_x^2 \varphi_{SS} + \lambda_1 \right). \end{aligned} \tag{14}$$

Eq. (12) can be written down in the independent variables (x, t)

$$iQ_t + \frac{1}{2}D_2(t)Q_{xx} + \sigma R_2(t)\left|Q\right|^2 Q - U(x,t)Q + i\widetilde{V'}Q_x = i\Gamma(t)Q. \qquad (15)$$

Let us transform Eq. (15) into the more convenient form

$$iQ_t + \frac{1}{2}D_2Q_{xx} + \sigma R_2\left|Q\right|^2 Q - UQ = i\Gamma Q \qquad (16)$$

by using the following condition

$$\widetilde{V'} = \frac{1}{2}D_2S_x\varphi_S + V - \lambda_1 S/S_x = 0. \qquad (17)$$

If we apply a reduction $V = -ia_1 = 0$ accepted in the IST method [9], then Eq. (17) defines a parameter λ_1

$$\lambda_1 = \frac{1}{2}D_2S_x^2\varphi_S/S, \qquad (18)$$

under which the real potential (13) is given by

$$U(S,T) = -2\gamma_0 + 2\lambda_0 S/S_x + \frac{1}{2}\left(\varphi_T + \varphi_S S_t\right) - \frac{1}{8}D_2S_x^2\varphi_S^2 \qquad (19)$$

and the gain or absorption coefficient (14) is represented by

$$\Gamma = \frac{1}{2}\frac{W(R_2,D_2)}{R_2D_2} - \frac{1}{4}D_2S_x^2\varphi_{SS} + \frac{1}{2}D_2S_x^2\varphi_S/S. \qquad (20)$$

Let us consider some special choices of variables to specify the solutions of (16). We assume that variables are factorized in the phase $\varphi(S,T)$

$$\varphi = C(T)S^\alpha$$

The first term in the real potential (19) represents some additional time-dependent phase $e^{2\gamma_0(t)t}$ of the solution $Q(x,t)$ for the equation (16) and, without loss of the generality, $\gamma_0 = 0$. The second term in (19) depends linearly on S. The NLSE with the linear spatial potential and constant λ_0, describing the case of Alfen waves, has been studied in Ref. [12]. We will study the more general case of chirped solitons in Part 4 of this article. Now, taking into account three last terms in (19), we obtain

$$U(S,T) = 2\lambda_0 S/S_x + \frac{1}{2}C_T S^\alpha + \frac{1}{2}\alpha C S^{\alpha-1}S_t - \frac{1}{8}D_2C^2S_x^2\alpha^2S^{2\alpha-2}. \qquad (21)$$

The gain or absorption coefficient (14) becomes

$$\Gamma(T) = \frac{1}{2}\frac{W(R_2,D_2)}{R_2D_2} + \frac{\alpha}{4}(3-\alpha)D_2S_x^2CS^{\alpha-2}. \qquad (22)$$

It is assumed here that two functions $\Gamma(T)$ and $\lambda_1(T)$ depend only on T and do not depend on S, thus we conclude that $\alpha = 0$ or $\alpha = 2$.

Let us find the solutions of Eq. (16) with time and space phase modulation (chirp) in the case $\alpha = 2$.

$$\varphi(S,T) = C(T)S^2.$$

In this case, Eq. (18) becomes

$$\lambda_1 = D_2 S_x^2 C.$$

Now, the real spatial-temporal potential (21) takes the form

$$U\left[S(x,t),T)\right] = 2\lambda_0 S/S_x + \tfrac{1}{2}\left(C_T - D_2 S_x^2 C^2\right) S^2 + CSS_t$$

Let us consider the simplest option to choose the variable $S(x,t)$ when the variables (x,t) are factorized: $S(x,t) = P(t)x$. In this case, all main characteristic functions: the phase modulation

$$\varphi(x,t) = \Theta(t)x^2, \tag{23}$$

the real potential

$$\begin{aligned} U(x,t) &= 2\lambda_0 x + \frac{1}{2}P^2\left(C_t - D_2P^2C^2 + 2\frac{P_t}{P}C\right)x^2 \\ &= 2\lambda_0 x + \frac{1}{2}\left(\Theta_t - D_2\Theta^2\right)x^2 \equiv 2\lambda_0(t)x + \frac{1}{2}\Omega^2(t)x^2, \end{aligned} \tag{24}$$

the gain (or absorption) coefficient

$$\Gamma(t) = \frac{1}{2}\left(\frac{W(R_2,D_2)}{R_2D_2} + D_2P^2C\right) = \frac{1}{2}\left(\frac{W(R_2,D_2)}{R_2D_2} + D_2\Theta\right) \tag{25}$$

and the spectral parameter λ_1

$$\lambda_1(t) = D_2P^2C = D_2(t)\Theta(t) \tag{26}$$

are given by expressions (23–26) dependent on the self-induced soliton phase shift $\Theta(t)$. In Eq. (24) a notation $\Omega^2(t) \equiv \Theta_t - D_2\Theta^2$ has been introduced.

Now we can rewrite the generalized NLSE (16) with time-dependent nonlinearity, dispersion and gain or absorption in the form of the nonautonomous NLSE with linear and parabolic potentials

$$iQ_t + \frac{1}{2}D_2(t)Q_{xx} + \sigma R_2(t)\left|Q\right|^2 Q - \frac{1}{2}\ \Omega^2(t)x^2Q = i\Gamma Q. \tag{27}$$

Substituting the phase profile $\Theta(t)$ from (25) into (24), it is straightforward to verify that the frequency of the harmonic potential $\Omega^2(t)$ is related with dispersion $D_2(t)$, nonlinearity $R_2(t)$ and gain or absorption coefficient $\Gamma(t)$ by the following conditions

$$\begin{aligned} \Omega^2(t)D_2(t) &= D_2(t)\frac{d}{dt}\left(\frac{\Gamma(t)}{D_2(t)}\right) - \Gamma^2(t) - \frac{d}{dt}\left(\frac{W(R_2,D_2)}{R_2D_2}\right) \\ &\quad + \left(2\Gamma(t) + \frac{d}{dt}\ln R_2(t)\right)\frac{W(R_2,D_2)}{R_2D_2} \qquad (28) \\ &= D_2(t)\frac{d}{dt}\left(\frac{\Gamma(t)}{D_2(t)}\right) - \Gamma^2(t) \\ &\quad + \left(2\Gamma(t) + \frac{d}{dt}\ln R_2(t)\right)\frac{d}{dt}\ln\frac{D_2(t)}{R_2(t)} - \frac{d^2}{dt^2}\ln\frac{D_2(t)}{R_2(t)}, \qquad (29) \end{aligned}$$

where $W(R_2,D) = R_2D_{2t}^{'} - D_2R_{2t}^{'}$ is the Wronskian.

After the substitutions

$$Q(x,t) = q(x,t)\exp\left[\int_0^t \Gamma(\tau)d\tau\right],$$
$$R(t) = R_2(t)\exp\left[2\int_0^t \gamma(\tau)d\tau\right],$$
$$D(t) = D_2(t),$$

Eq. (27) can be transformed to the generalized NLSE without gain or loss term

$$i\frac{\partial q}{\partial t} + \frac{1}{2}D(t)\frac{\partial^2 q}{\partial x^2} + \left[\sigma R(t)\left|q\right|^2 - 2\lambda_0(t)x - \frac{1}{2}\Omega^2(t)x^2\right]q = 0. \tag{30}$$

Finally, the Lax equation (1) with matrices (2–5) provides the nonautonomous model (30) under condition that dispersion $D(t)$, nonlinearity $R(t)$, and the harmonic potential satisfy to the following exact integrability conditions

$$\Omega^2(t)D(t) = \frac{W(R,D)}{RD}\frac{d}{dt}\ln R(t) - \frac{d}{dt}\left(\frac{W(R,D)}{RD}\right)$$
$$= \frac{d}{dt}\ln D(t)\frac{d}{dt}\ln R(t) - \frac{d^2}{dt^2}\ln D(t) - R(t)\frac{d^2}{dt^2}\frac{1}{R(t)}. \tag{31}$$

The self-induced soliton phase shift is given by

$$\Theta(t) = -\frac{W\left[(R(t), D(t)\right]}{D^2(t)R(t)} \tag{32}$$

and the time-dependent spectral parameter is represented by

$$\Lambda(t) = \kappa(t) + i\eta(t) = \frac{D_0 R(t)}{R_0 D(t)}\left[\Lambda(0) + \frac{R_0}{D_0}\int\limits_0^t \frac{\lambda_0(\tau)D(\tau)}{R(\tau)}d\tau\right], \tag{33}$$

where the main parameters: time invariant eigenvalue $\Lambda(0) = \kappa_0 + i\eta_0$; $D_0 = D(0)$; $R_0 = R(0)$ are defined by the initial conditions. We call Eq. (31) as the law of the soliton adaptation to the external potentials.

The basic property of classical solitons, to interact elastically, holds true, but the novel feature of the nonautonomous solitons arises. Namely, both amplitudes and speeds of the solitons, and consequently, their spectra, during the propagation and after the interaction are no longer the same as those prior to the interaction. All nonautonomous solitons generally move with varying amplitudes $\eta(t)$ and speeds $\kappa(t)$ adapted both to the external potentials and to the dispersion $D(t)$ and nonlinearity $R(t)$ changes.

Having obtained the eigenvalue equations for scattering potential, we can write down the general solutions for bright ($\sigma = +1$) and dark ($\sigma = -1$) nonautonomous solitons applying the auto-Bäcklund transformation [28] and the recurrent relation

$$q_n(x,t) = -q_{n-1}(x,t) - \frac{4\eta_n \widetilde{\mathbf{\Gamma}}_{n-1}(x,t)}{1+\left|\widetilde{\mathbf{\Gamma}}_{n-1}(x,t)\right|^2}\sqrt{\frac{D(t)}{R(t)}}\exp[-i\Theta x^2/2], \tag{34}$$

which connects the $(n-1)$ and n-soliton solutions by means of the so-called pseudo-potential $\widetilde{\mathbf{\Gamma}}_{n-1}(x,t) = \psi_1(x,t)/\psi_2(x,t)$ for the $(n-1)$-soliton scattering functions $\psi(x,t) = (\psi_1\psi_2)^T$.

Bright $q_1^+(x,t)$ and dark $q_1^-(x,t)$ soliton solutions are represented by the following analytic expressions:

$$q_1^+(x,t \mid \sigma = +1) = 2\eta_1(t)\sqrt{\frac{D(t)}{R(t)}}\operatorname{sech}\left[\xi_1(x,t)\right] \times \exp\left\{-i\left(\frac{\Theta(t)}{2}x^2 + \chi_1(x,t)\right)\right\}; \tag{35}$$

$$q_1^-(x,t \mid \sigma = -1) = 2\eta_1(t)\sqrt{\frac{D(t)}{R(t)}}\left[\sqrt{(1-a^2)} + ia\tanh\zeta\,(x,t)\right] \times \exp\left\{-i\left(\frac{\Theta(t)}{2}x^2 + \phi(x,t)\right)\right\}, \tag{36}$$

$$\zeta(x,t) = 2a\eta_1(t)x + 4a\int_0^t D(\tau)\eta_1(\tau)\kappa_1(\tau)d\tau, \tag{37}$$

$$\phi(x,t) = 2\left[\kappa_1(t) - \eta_1(t)\sqrt{(1-a^2)}\right]x + 2\int_0^t D(\tau)\left[\kappa_1^2 + \eta_1^2\left(3-a^2\right) - 2\kappa_1\eta_1\sqrt{(1-a^2)}\right]d\tau. \tag{38}$$

Dark soliton Eq. (36) has an additional parameter, $0 \le a \le 1$, which designates the depth of modulation (the blackness of gray soliton) and its velocity against the background. When $a = 1$, dark soliton becomes black. For optical applications, Eq. (36) can be easily transformed into the Hasegawa and Tappert form for the nonautonomous dark solitons [2] under the condition $\kappa_0 = \eta_0\sqrt{(1-a^2)}$ that corresponds to the special choice of the retarded frame associated with the group velocity of the soliton

$$q_1^-(x,t \mid \sigma = -1) = 2\eta_1(t)\sqrt{\frac{D(t)}{R(t)}}\left[\sqrt{(1-a^2)} + ia\tanh\widetilde{\zeta}\,(x,t)\right] \times \exp\left\{-i\left(\frac{\Theta(t)}{2}x^2 + \widetilde{\phi}(x,t)\right)\right\},$$

$$\widetilde{\zeta}(x,t) = 2a\eta_1(t)x + 4a\int_0^t D(\tau)\eta_1(\tau)\left[\eta_1(\tau)\sqrt{(1-a^2)} + K(\tau)\right]d\tau,$$

$$\widetilde{\phi}(x,t) = 2K(t)x + 2\int_0^t D(\tau)\left[K^2(\tau) + 2\eta_1^2(\tau)\right]d\tau,$$

$$K(t) = \frac{R(t)}{D(t)}\int_0^t \lambda_0(\tau)\frac{D(\tau)}{R(\tau)}d\tau.$$

Notice that the solutions considered here hold only when the nonlinearity, dispersion and confining harmonic potential are related by Eq. (31), and both $D(t) \neq 0$ and $R(t) \neq 0$ for all times by definition.

Two-soliton $q_2(x,t)$ solution for $\sigma = +1$ follows from Eq. (34)

$$q_2(x,t) = 4\sqrt{\frac{D(t)}{R(t)}}\frac{\mathrm{N}\ (x,t)}{\mathrm{D}\ (x,t)}\exp\left[-\frac{i}{2}\Theta(t)x^2\right], \tag{39}$$

where the numerator N (x,t) is given by

$$\begin{aligned}\mathrm{N} = \ & \cosh\xi_2 \exp(-i\chi_1) \\ & \quad \times [(\kappa_2-\kappa_1)^2 + 2i\eta_2(\kappa_2-\kappa_1)\tanh\xi_2 + \eta_1^2 - \eta_2^2] \\ & + \eta_2\cosh\xi_1\exp(-i\chi_2) \\ & \quad \times [(\kappa_2-\kappa_1)^2 - 2i\eta_1(\kappa_2-\kappa_1)\tanh\xi_1 - \eta_1^2 + \eta_2^2],\end{aligned} \tag{40}$$

and the denominator D (x,t) is represented by

$$\begin{aligned}\mathrm{D} = \ & \cosh(\xi_1+\xi_2)\left[(\kappa_2-\kappa_1)^2 + (\eta_2-\eta_1)^2\right] \\ & + \cosh(\xi_1-\xi_2)\left[(\kappa_2-\kappa_1)^2 + (\eta_2+\eta_1)^2\right] \\ & - 4\eta_1\eta_2\cos(\chi_2-\chi_1).\end{aligned} \tag{41}$$

Arguments and phases in Eqs. (39–41)

$$\xi_i(x,t) = 2\eta_i(t)x + 4\int_0^t D(\tau)\eta_i(\tau)\kappa_i(\tau)d\tau, \tag{42}$$

$$\chi_i(x,t) = 2\kappa_i(t)x + 2\int_0^t D(\tau)\left[\kappa_i^2(\tau) - \eta_i^2(\tau)\right]d\tau \tag{43}$$

are related with the amplitudes

$$\eta_i(t) = \frac{D_0 R(t)}{R_0 D(t)}\eta_{0i}, \tag{44}$$

and velocities

$$\kappa_i(t) = \frac{D_0 R(t)}{R_0 D(t)} \left[\kappa_{0i} + \frac{R_0}{D_0} \int_0^t \frac{\lambda_0(\tau) D(\tau)}{R(\tau)} d\tau \right] \tag{45}$$

of the nonautonomous solitons, where κ_{0i} and η_{0i} correspond to the initial velocity and amplitude of the ith soliton $(i = 1, 2)$.

Eqs. (39–45) describe the dynamics of two bounded solitons at all times and all locations. Obviously, these soliton solutions reduce to classical soliton solutions in the limit of autonomous nonlinear and dispersive systems given by conditions: $R(t) = D(t) = 1$, and $\lambda_0(t) = \Omega(t) \equiv 0$ for canonical NLSE without external potentials.

4. Chirped optical solitons with moving spectra in nonautonomous systems: colored nonautonomous solitons

Both the nonlinear Schrödinger equations (27), (30) and the Lax pair equations (1–5) are written down here in the most general form. The transition to the problems of optical solitons is accomplished by the substitution $x \to T$ (or $x \to X$); $t \to Z$ and $q^+(x,t) \to \widetilde{u}^+(Z, T(\ or\ X))$ for bright solitons, and $[q^-(x,t)]^* \to \widetilde{u}^-(Z, T(\ or\ X))$ for dark solitons, where the asterisk denotes the complex conjugate, Z is the normalized distance, and T is the retarded time for temporal solitons, while X is the transverse coordinate for spatial solitons.

The important special case of Eq. (30) arises under condition $\Omega^2(Z) = 0$. Let us rewrite Eq. (30) by using the reduction $\Omega = 0$, which denotes that the confining harmonic potential is vanishing

$$i\frac{\partial u}{\partial Z} + \frac{\sigma}{2} D(Z) \frac{\partial^2 u}{\partial T^2} + R(Z) |u|^2 u - 2\sigma\alpha(Z) T u = 0. \tag{46}$$

This implies that the self-induced soliton phase shift $\Theta(Z)$, dispersion $D(Z)$, and nonlinearity $R(Z)$ are related by the following law of soliton adaptation to external linear potential

$$\frac{D(Z)}{D_0} = \frac{R(Z)}{R_0} \exp\left\{ -\frac{\Theta_0 D_0}{R_0} \int_0^Z R(\tau) d\tau \right\}. \tag{47}$$

Nonautonomous exactly integrable NLSE model given by Eqs. (46, 47) can be considered as the generalization of the well-studied Chen and Liu model [12] with linear potential $\lambda_0(Z) = \alpha_0 = const$ and $D(Z) = D_0 = R(Z) = R_0 = 1$, $\sigma = +1$, $\Theta_0 = 0$. We stress that the accelerated solitons predicted by Chen and Liu in plasma have been discovered in nonlinear fiber optics only decade later $[3 - -5]$. Notice that nonautonomous solitons with nontrivial self-induced phase shifts and varying amplitudes, speeds and spectra for Eq. (46) are given in quadratures by Eqs. (35–38) under condition $\Omega^2(Z) = 0$.

Let us show that the so-called Raman colored optical solitons can be approximated by this equation. Self-induced Raman effect (also called as soliton self-frequency shift) is being described by an additional term in the NLSE: $-\sigma_R U\partial \mid U \mid^2 /\partial T$ where σ_R originates from the frequency-dependent Raman gain $[3-5]$. Assuming that soliton amplitude does not vary significantly during self-scattering $\mid U \mid^2 = \eta^2 \mathrm{sech}^2(\eta T)$, we obtain that

$$\sigma_R \frac{\partial \mid U \mid^2}{\partial T} \approx -2\sigma_R \eta^4 T = 2\alpha_0 T$$

and $dv/dZ = \sigma_R \eta^4/2$, where $v = \kappa/2$. The result of soliton perturbation theory $[3-5]$ gives $dv/dZ = 8\sigma_R \eta^4/15$. This fact explains the remarkable stability of colored Raman solitons that is guaranteed by the property of the exact integrability of the Chen and Liu model [12]. More general model Eq. (46) and its exact soliton solutions open the possibility of designing an effective soliton compressor, for example, by drawing a fiber with $R(Z) = 1$ and $D(Z) = \exp(-c_0 Z)$,where $c_0 = \Theta_0 D_0$. It seems very attractive to use the results of nonautonomous solitons concept in ultrashort photonic applications and soliton lasers design.

Another interesting feature of the novel solitons, which we called colored nonautonomous solitons here, is associated with the nontrivial dynamics of their spectra. Frequency spectrum of the chirped nonautonomous optical soliton moves in the frequency domain. In particular, if dispersion and nonlinearity evolve in unison $D(t) = R(t)$ or $D = R = 1$, the solitons propagate with identical spectra but with totally different time-space behavior.

Consider in more details the case when the nonlinearity $R = R_0$ stays constant but the dispersion varies exponentially along the propagation distance

$$D(Z) = D_0 \exp\left(-c_0 Z\right), \qquad \Theta(Z) = \Theta_0 \exp\left(c_0 Z\right).$$

Let us obtain the one and two soliton solutions in this case with the lineal potential that, for simplicity, does not depend on time: $\lambda_0(Z) = \alpha_0 = \mathrm{const}$

$$U_1(Z,T) = 2\eta_{01}\sqrt{D_0 \exp\left(c_0 Z\right)}\mathrm{sech}\left[\xi_1(Z,T)\right] \times \exp\left[-\frac{i}{2}\Theta_0 \exp\left(c_0 Z\right) T^2 - i\chi_1(Z,T)\right], \tag{48}$$

$$U_2(Z,T) = 4\sqrt{D_0 \exp\left(-c_0 Z\right)}\frac{\mathrm{N}\ (Z,T)}{\mathrm{D}\ (Z,T)} \exp\left[-\frac{i}{2}\Theta_0 \exp\left(c_0 Z\right) T^2\right], \tag{49}$$

where the nominator $\mathrm{N}\ (Z,T)$ and denominator $\mathrm{D}\ (Z,T)$ are given by Eqs. (40, 41) and

$$\xi_i(Z,T) = 2\eta_{0i} T \exp\left(c_0 Z\right) + 4D_0\eta_{0i} \times \left\{\frac{\kappa_{0i}}{c_0}\left[\exp\left(c_0 Z\right) - 1\right] + \frac{\alpha_0}{c_0}\left[\frac{\exp\left(c_0 Z\right) - 1}{c_0} - Z\right]\right\}, \tag{50}$$

$$\chi_i(Z,T) = 2\kappa_{0i} T \exp(c_0 Z) + 2D_0\left(\kappa_{0i}^2 - \eta_{0i}^2\right)\frac{\exp(2c_0 Z) - 1}{2c_0}$$
$$+ 2T\frac{\alpha_0}{c_0}\left[\exp(c_0 Z) - 1\right] + 4D_0\kappa_{0i}\frac{\alpha_0}{c_0}\left[\frac{\exp(c_0 Z) - 1}{c_0} - t\right]$$
$$+ 2D_0\left(\frac{\alpha_0}{c_0}\right)^2\left[\frac{\exp(c_0 Z) - \exp(-c_0 Z)}{c_0} - 2Z\right]. \quad (51)$$

The initial velocity and amplitude of the ith soliton ($i = 1, 2$) are denoted by κ_{0i} and η_{0i}.

The limit case of the Eqs. (48–51) appears when $c_0 \to \infty$ (that means $D(Z) = D_0$ =constant) and corresponds to the Chen and Liu model [12]. The solitons with argument and phase

$$\xi(Z,T) = 2\eta_0\left(T + 2\kappa_0 Z + \alpha_0 Z^2 - T_0\right),$$
$$\chi(Z,T) = 2\kappa_0 T + 2\alpha_0 T Z + 2\left(\kappa_0^2 - \eta_0^2\right) Z + 2\kappa_0\alpha_0 Z^2 + \frac{2}{3}\alpha_0^2 Z^3$$

represents the particle-like solutions which may be accelerated and reflected from the lineal potential.

5. Bound states of colored nonautonomous optical solitons. Comparison of the Satsuma-Yajima canonical breather with nonautonomous "agitated" breather

Let us now give the explicit formula of the soliton solutions for the case where all eigenvalues are pure imaginary, or the initial velocities of the solitons are equal to zero. In the case $N = 1$ and $\lambda_0(Z) = 0$, and we obtain

$$U_1(Z,T) = 2\eta_{01}\sqrt{D_0 \exp(c_0 Z)}\,\text{sech}\left[2\eta_{01} T \exp(c_0 Z)\right]$$
$$\times \exp\left[-\frac{i}{2}\Theta_0 \exp(c_0 Z)\, T^2 + i2D_0\eta_{01}^2\frac{\exp(2c_0 Z) - 1}{2c_0}\right]. \quad (52)$$

This result shows that the laws of soliton adaptation in external potentials Eq. (31) allow to stabilize the soliton even without a trapping potential. In addition, Eq. (52) indicates the possibility for the optimal compression of solitons. We stress that direct computer experiment confirms the exponential in time soliton compression scenario in full accordance with analytical expression Eq. (52).

The bound two-soliton solution for the case of the pure imaginary eigenvalues is represented by

$$U_2(Z,T) = 4\sqrt{D_0 \exp(-c_0 Z)}\frac{\mathrm{N}\ (Z,T)}{\mathrm{D}\ (Z,T)}\exp\left[-\frac{i}{2}\Theta_0 \exp(c_0 Z)\, T^2\right], \quad (53)$$

where

$$\mathrm{N} = \left(\eta_{01}^2 - \eta_{02}^2\right)\exp(c_0 Z)\left[\eta_{01}\cosh\xi_2 \exp(-i\chi_1) - \eta_{02}\cosh\xi_1 \exp(-i\chi_2)\right], \quad (54)$$

$$\begin{aligned} \mathrm{D} = & \cosh(\xi_1+\xi_2)\left(\eta_{01}-\eta_{02}\right)^2+\cosh(\xi_1-\xi_2)\left(\eta_{01}+\eta_{02}\right)^2 \\ & -4\eta_{01}\eta_{02}\cos\left(\chi_2-\chi_1\right), \end{aligned} \tag{55}$$

and

$$\xi_i(Z,T) = 2\eta_{0i}T\exp\left(c_0 Z\right), \tag{56}$$

$$\chi_i(Z,T) = -2D_0\eta_{0i}^2\frac{\exp\left(2c_0 Z\right)-1}{2c_0}+\chi_{i0}. \tag{57}$$

For the particular case of $\eta_{10}=1/2$, $\eta_{20}=3/2$ Eqs. (53–57) transform to

$$\begin{aligned} U_2(Z,T) = & 4\sqrt{D_0\exp\left(-c_0 Z\right)}\exp\left[-\frac{i}{2}\Theta_0\exp\left(c_0 Z\right)T^2\right] \\ & \times\exp\left[\frac{i}{4c_0}D_0\left[\exp\left(2c_0 Z\right)-1\right]+\chi_{10}\right] \\ & \times\frac{\cosh 3X-3\cosh X\exp\left\{i2D_0\left[\exp\left(2c_0 Z\right)-1\right]/c_0+i\Delta\varphi\right\}}{\cosh 4X+4\cosh 2X-3\cos\left\{2D_0\left[\exp\left(2c_0 Z\right)-1\right]/c_0+\Delta\varphi\right\}}, \end{aligned} \tag{58}$$

where $X=T\exp(c_0 Z)$, $\Delta\varphi=\chi_{20}-\chi_{10}$.

In the limit $D(Z)=D_0=1$ and $c_0=0$ this solution reduces to the well-known breather solution, which was found by Satsuma and Yajima [29] and was called as the Satsuma-Yajima breather:

$$U_2(Z,T) = 4\frac{\cosh 3T+3\cosh T\exp\left(4iZ\right)}{\cosh 4T+4\cosh 2T+3\cos 4Z}\exp\left(\frac{iZ}{2}\right).$$

At $Z=0$ it takes the simple form $U(Z,T)=2\,\mathrm{sech}(T)$. An interesting property of this solution is that its form oscillates with the so-called soliton period $T_{\mathrm{sol}}=\pi/2$.

According to Eq. (58), the soliton period becomes dependent on time. We stress that the Satsuma and Yajima breather solution can be obtained from the general solution if and only if the soliton phases are chosen properly, precisely when $\Delta\varphi=\pi$. The intensity profiles of the wave build up a complex landscape of peaks and valleys and reach their peaks at the points of the maximum. Decreasing group velocity dispersion (or increasing nonlinearity) stimulates the Satsuma-Yajima breather to accelerate its period of "breathing" and to increase its peak amplitudes of "breathing", that is why we call this effect as "agitated breather" in nonautonomous system.

6. Designable integrability of the variable coefficient nonlinear Schrödinger equation

Recently, Jingsong He and Yishen Li have found the possibility to solve the more general NLSE with varying both in time and space coefficients and with arbitrary potential $V(x,t)$

$$i\frac{\partial}{\partial t}\psi(x,t)+\frac{1}{2}D(x,t)\frac{\partial^2}{\partial x^2}\psi(x,t)+R(x,t)|\psi(x,t)|^2\psi(x,t)-V(x,t)\psi(x,t)=0, \tag{59}$$

where $D(x,t)$, $V(x,t)$, $R(x,t)$ are three real functions of x and t. Specifically, the transformation

$$\psi(x,t) = q(X,T)p(x,t)e^{i\phi(x,t)} \tag{60}$$

maps Eq. (59) into the canonical NLSE with constant coefficients

$$i\frac{\partial q}{\partial T} + \frac{1}{2}\frac{\partial^2 q}{\partial X^2} + |q|^2 q = 0. \tag{61}$$

Meanwhile, coefficients $D(x,t)$, $V(x,t)$, $R(x,t)$ are given analytically with several arbitrary functions appearing in the transformation. Thus we can design $D(x,t)$, $V(x,t)$, $R(x,t)$ according to different physical considerations by means of the selections of the arbitrary functions so that the integrability is guaranteed. Therefore, we show that the variable coefficients NLSE (VC NLSE) possesses the designable integrability (DI), which originates from the rigid integrability of the NLSE and the transformation Eq. (60).

Proposition. *For the five real smooth functions $c_1(t), c_2(t), c_3(t), T(t), F(x)$, set $F(x)c_1(t) > 0$, if*

$$D(x,t) = \frac{T_t}{F(x)^2 c_1(t)^2}, \quad R(x,t) = \frac{F(x)T_t}{c_1(t)}, \tag{62}$$

$$V(x,t) = -\frac{1}{8}\frac{-3T_tF_x^2 + 2T_tF_{xx}F + 8c_1(t)^2\phi_tF^4 + 4T_t(\phi_x)^2F^2}{c_1(t)^2F(x)^4}, \tag{63}$$

then the following transformation

$$\psi(x,t) = q(X,T)p(x,t)e^{i\phi(x,t)} \tag{64}$$

maps Eq. (59) *to the standard NLSE Eq.* (61)*. Here*

$$\begin{aligned} X &= X(x,t) \\ &= \int F(x)c_1(t)dx + c_3(t), \quad T = T(t), \end{aligned} \tag{65}$$

$$\begin{aligned} p &= p(x,t) \\ &= \frac{c_1(t)}{\sqrt{F(x)c_1(t)}}, \end{aligned} \tag{66}$$

$$\begin{aligned} \phi &= \phi(x,t) \\ &= -\int \frac{\left(\int F(x)c_{1t}dx + c_{3t}\right)F(x)c_1(t)}{T_t}dx + c_2(t) \end{aligned} \tag{67}$$

To illustrate the wide applicability of our methodology, and motivated by the importance of the external potentials in the BEC and nonlinear optics systems, we designed two kinds of integrable VC NLSE with optical super-lattice potentials (or periodic potentials) and multi-well potentials, respectively [30].

7. Conclusions

In summary, the solution technique based on the generalized Lax pair operator method opens the possibility to study in details the nonlinear dynamics of solitons in nonautonomous nonlinear and dispersive physical systems. We have focused on the situation in which the generalized nonautonomous NLSE model was found to be exactly integrable from the point of view of the inverse scattering transform method. We have derived the laws of a soliton adaptation to the external potential. It is precisely this soliton adaptation mechanism which was of prime physical interest in our paper. We clarified some examples in order to gain a better understanding into this physical mechanism which can be considered as the interplay between nontrivial time-dependent parabolic soliton phase and external time-dependent potential. We stress that this nontrivial time-space dependent phase profile of nonautonomous soliton depends on the Wronskian of nonlinearity $R(t)$ and dispersion $D(t)$ and this profile does not exist for canonical NLSE soliton when $R(t) = D(t) = 1$. Nonautonomous solitons trapped inside the harmonic oscillator potential can form novel kind of colored solitons with periodically varying (along the propagation distance Z) average frequency shifted into the "red" and "blue" spectral regions for the case of optical solitons. For the case of trapped matter wave solitons, their average wave numbers K(T) periodically oscillate in time T. We have studied the main features of nonautonomous optical and matter-wave solitons, their bound states formation and the transformation of the Satsuma-Yajima breather into "agitated" nonautonomous breather. Novel method for solution of the variable coefficient nonlinear Schrödinger equation (called as designable integrability) is also considered.

We would like to conclude by saying that the concept of adaptation is of primary importance in nature and nonautonomous solitons that interact elastically and generally move with varying amplitudes, speeds, and spectra adapted both to the external potentials and to the dispersion and nonlinearity changes can be fundamental objects of nonlinear science.

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Asymptotics of Individual Eigenvalues of a Class of Large Hessenberg Toeplitz Matrices

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Abstract. We study the asymptotic behavior of individual eigenvalues of the n-by-n truncations of certain infinite Hessenberg Toeplitz matrices as n goes to infinity. The generating function of the Toeplitz matrices is supposed to be of the form $a(t) = t^{-1}(1-t)^{\alpha} f(t)$ $(t \in \mathbb{T})$, where α is a positive real number but not an integer and f is a smooth function in H^{∞}. The classes of generating functions considered here and in a recent paper by Dai, Geary, and Kadanoff are overlapping, and in the overlapping cases, our results imply in particular a rigorous justification of an asymptotic formula which was conjectured by Dai, Geary, and Kadanoff on the basis of numerical computations.

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Keywords. Toeplitz matrix, eigenvalue, Fourier integral, asymptotic expansion.

1. Introduction and main results

The $n \times n$ Toeplitz matrix generated by a complex-valued function $a \in L^1$ on the unit circle $\mathbb{T}$ is the matrix $T_n(a) = \left(a_{j-k}\right)_{j,k=0}^{n-1}$, where a_k is the kth Fourier coefficient of the function a, that is, $a_k = \int_0^{2\pi} a\left(e^{i\theta}\right)e^{-ik\theta}\,d\theta/2\pi$, $k \in \mathbb{Z}$. The function a is referred to as the symbol of the matrices $T_n(a)$.

If a is real-valued, then the matrices $T_n(a)$ are all Hermitian, and in this case a number of results on the asymptotics of the eigenvalues of $T_n(a)$ are known; see, for example, [6], [7], [13], [16], [18], [20], [21], [23], [24], [26], [28], [29], [30]. We here consider genuinely complex-valued symbols, in which case the overall picture is less complete. Papers [12], [15], [19] describe the limiting behavior of the eigenvalues of $T_n(a)$ if a is a rational function, while papers [1] and [27] are devoted to the asymptotic eigenvalue distribution in the case of non-smooth symbols. In [25] and

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[27], it is in particular shown that if $a \in L^\infty$ and the essential range $\mathcal{R}(a)$ does not separate the plane, then the eigenvalues of $T_n(a)$ approximate $\mathcal{R}(a)$. Many of the results of the papers cited above can also be found in the books [5], [8], [9].

Throughout what follows we assume that a is a complex-valued continuous function on $\mathbb{T}$. In that case $\mathcal{R}(a) = a(\mathbb{T})$. When the eigenvalues of $T_n(a)$ approach $\mathcal{R}(a)$ asymptotically in the sense that

$$\lim_{n\to\infty} \frac{\operatorname{trace}\varphi\big(T_n(a)\big)}{n} = \int_0^{2\pi} \varphi\big(a(e^{i\theta})\big)\,\frac{d\theta}{2\pi} \tag{1.1}$$

for a sufficiently rich supply of test functions φ, one says that they have canonical distribution. In 1990, Widom [27] showed that if $\mathcal{R}(a)$ is a Jordan curve and a is smooth on $\mathbb{T}$ minus a single point but not smooth on all of $\mathbb{T}$, then the spectrum of $T_n(a)$ has canonical distribution. He also raised the following intriguing conjecture, which is still an open problem:

The eigenvalues of $T_n(a)$ are canonically distributed except when a extends analytically to an annulus $r < |z| < 1$ or $1 < |z| < R$.

Results like (1.1) or of the type that the spectrum of $T_n(a)$ converges to some limiting set in the Hausdorff metric do not provide us with information on the asymptotic behavior of individual eigenvalues. The asymptotic behavior of the extreme eigenvalues of Hermitian Toeplitz matrices is fairly well understood; see the references cited above. Paper [6] contains asymptotic expansions for individual inner eigenvalues of certain banded Hermitian Toeplitz matrices. The recent papers [11] and [17] concern asymptotic formulas for individual eigenvalues of Toeplitz matrices whose symbols are complex-valued and have a so-called Fisher-Hartwig singularity. These are special symbols that are smooth on $\mathbb{T}$ minus a single point but not smooth on the entire circle $\mathbb{T}$; see [8], [9].

To be more specific, Dai, Geary, and Kadanoff [11] considered symbols of the form

$$a(t) = \left(2 - t - \frac{1}{t}\right)^{\gamma} (-t)^{\beta}, \quad t \in \mathbb{T},$$

where $0 < \gamma < -\beta < 1$. They conjectured that the eigenvalues $\lambda = \lambda_{j,n}$ satisfy

$$\lambda_{j,n} \approx a\left(n^{(2\gamma+1)/n} \exp\left(-\frac{2\pi i}{n} j\right)\right), \quad j = 0, \dots, n-1, \tag{1.2}$$

and confirmed this conjecture numerically.

Let H^∞ be the usual Hardy space of (boundary values of) bounded analytic functions in the unit disk $\mathbb{D}$. Given $a \in C(\mathbb{T})$, we denote by $\operatorname{wind}_\lambda(a)$ the winding number of a about a point $\lambda \in \mathbb{C} \setminus \mathcal{R}(a)$ and by $\mathcal{D}(a)$ the set of all $\lambda \in \mathbb{C}$ for which $\operatorname{wind}_\lambda(a) \neq 0$. In this paper we study the eigenvalues of $T_n(a)$ for symbols $a(t) = t^{-1}h(t)$ with the following properties:

1. $h \in H^\infty$ and $h_0 \neq 0$;
2. $h(t) = (1-t)^\alpha f(t)$, where $\alpha \in [0,\infty) \setminus \mathbb{Z}$ and $f \in C^\infty(\mathbb{T})$;

3. h has an analytic extension to an open neighborhood W of $\mathbb{T}\setminus\{1\}$ not containing the point 1;
4. $\mathcal{R}(a)$ is a Jordan curve in $\mathbb{C}$ and $\mathrm{wind}_\lambda(a) = -1$ for each $\lambda \in \mathcal{D}(a)$.

Here h_0 is the zeroth Fourier coefficient of h.

According to [27], in our case the spectrum of $T_n(a)$ has canonical distribution. Note that when $\beta = \gamma - 1$ and $f \equiv 1$, our symbol coincides with the one of [11].

Let $D_n(a)$ denote the determinant of $T_n(a)$. Thus, the eigenvalues λ of $T_n(a)$ are the solutions of the equation $D_n(a-\lambda) = 0$. Our assumptions imply that $T_n(a)$ is a Hessenberg matrix, that is, it arises from a lower triangular matrix by adding the super-diagonal. This circumstance together with the Baxter-Schmidt formula for Toeplitz determinants allows us to express $D_n(a-\lambda)$ as a Fourier integral. The value of this integral mainly depends on λ and on the singularity of $(1-t)^\alpha$ at the point 1. Let W_0 be a small open neighborhood of zero in $\mathbb{C}$. We show that for every point $\lambda \in \mathcal{D}(a) \cap \big(a(W) \setminus W_0\big)$ there exists a unique point $t_\lambda \notin \overline{\mathbb{D}}$ such that $a(t_\lambda) = \lambda$. After exploring the contributions of λ and the singular point 1 to the Fourier integral, we get the following asymptotic expansion for $D_n(a-\lambda)$.

Theorem 1.1. *Let $a(t) = t^{-1}h(t)$ be a symbol with properties 1 to 4. Then for every small open neighborhood W_0 of zero in $\mathbb{C}$ and every $\lambda \in \mathcal{D}(a) \cap \big(a(W) \setminus W_0\big)$,*

$$D_n(a-\lambda) = (-h_0)^{n+1}\left(\frac{1}{t_\lambda^{n+2}a'(t_\lambda)} - \frac{f(1)\Gamma(\alpha+1)\sin(\alpha\pi)}{\pi\lambda^2 n^{\alpha+1}} + R_1(n,\lambda)\right), \quad (1.3)$$

where $R_1(n,\lambda) = \mathcal{O}(1/n^{\alpha+\alpha_0+1})$ as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$. Here $\alpha_0 = \min\{\alpha, 1\}$.

The first term in brackets is the contribution of λ, while the second is the contribution of the point 1. In the case where f is identically 1, the previous Theorem is essentially already in [3].

Here now are our main results. Let W_0 be a small open neighborhood of the origin in $\mathbb{C}$ and put $\omega_n := \exp(-2\pi i/n)$. For each n there exist integers n_1 and n_2 such that $\omega_n^{n_1}, \omega_n^{n-n_2} \in a^{-1}(W_0)$ but $\omega_n^{n_1+1}, \omega_n^{n-n_2-1} \notin a^{-1}(W_0)$. Recall that $a(t_\lambda) = \lambda$.

Theorem 1.2. *Let $a(t) = t^{-1}h(t)$ be a symbol with properties 1 to 4. Then for every small open neighborhood W_0 of the origin in $\mathbb{C}$ and every j between n_1 and $n-n_2$,*

$$t_{\lambda_{j,n}} = n^{(\alpha+1)/n}\omega_n^j\left(1 + \frac{1}{n}\log\left(\frac{a^2(\omega_n^j)}{C_1 a'(\omega_n^j)\omega_n^{2j}}\right) + R_2(n,j)\right), \quad (1.4)$$

where $R_2(n,j) = \mathcal{O}(1/n^{\alpha_0+1}) + \mathcal{O}(\log n/n^2)$ as $n \to \infty$, uniformly with respect to j in $(n_1, n-n_2)$. Here $\alpha_0 = \min\{\alpha, 1\}$ and

$$C_1 = \frac{f(1)\Gamma(\alpha+1)\sin(\alpha\pi)}{\pi}.$$

Formula (1.4) proves conjecture (1.2) in the special case $\beta = \gamma - 1$. It shows that as n increases, the point $t_{\lambda_{j,n}}$ is close to $n^{(\alpha+1)/n}\omega_n^j$. Finally, we take the value of a at the point (1.4) to obtain the following expression for $\lambda_{j,n}$.

Theorem 1.3. *Let $a(t) = t^{-1}h(t)$ be a symbol with properties* 1 *to* 4. *Then for every small neighborhood W_0 of zero in $\mathbb{C}$ and every j between n_1 and $n - n_2$,*

$$\lambda_{j,n} = a(\omega_n^j) + (\alpha+1)\,\omega_n^j a'(\omega_n^j)\frac{\log n}{n} + \frac{\omega_n^j a'(\omega_n^j)}{n}\log\left(\frac{a^2(\omega_n^j)}{C_1 a'(\omega_n^j)\omega_n^{2j}}\right) + R_3(n,j), \tag{1.5}$$

where C_1 is as in Theorem 1.2 *and $R_3(n,j) = \mathcal{O}(1/n^{\alpha_0+1}) + \mathcal{O}(\log^2 n/n^2)$ as $n \to \infty$, uniformly with respect to j in $(n_1, n-n_2)$.*

The idea to use just the singularity $(1-t)^\alpha$ in order to study phenomena connected with eigenvalue asymptotics was also employed in [4].

We remark that we wrote down only the first few terms in our asymptotic expansions but that our method is constructive and would allow us to get as many terms as we desire. Clearly, conjecture (1.2) corresponds to the first term in our asymptotic expansion (1.4). Figure 1 illustrates Theorem 1.3. In the last section, we present another simulation graphic and error tables made with *Matlab* software to show that incorporating the second term of our expansion (1.4) (= third term in (1.5)) reduces the error to nearly one tenth.

2. Toeplitz determinant

Lemma 2.1. *Let $a(t) = t^{-1}h(t)$ have properties* 1 *and* 4. *Then, for each $\lambda \in \mathcal{D}(a)$ and every $n \in \mathbb{N}$, and with $[\ ]_n$ denoting the nth Fourier coefficient,*

$$D_n(a-\lambda) = (-1)^n h_0^{n+1}\left[\frac{1}{h(t)-\lambda t}\right]_n. \tag{2.1}$$

Proof. This can be deduced from the Baxter-Schmidt formula [2], which is also in [5, p. 37]. For the reader's convenience, we include a direct proof of (2.1). Obviously,

$$T_{n+1}(h-\lambda t) = \left[\begin{array}{ccccc|c} h_0 & 0 & 0 & \cdots & 0 & 0 \\ \hline h_1-\lambda & h_0 & 0 & \cdots & 0 & 0 \\ h_2 & h_1-\lambda & h_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \cdots & h_0 & 0 \\ h_n & h_{n-1} & h_{n-2} & \cdots & h_1-\lambda & h_0 \end{array}\right] \tag{2.2}$$

and

$$T_n(a-\lambda) = \begin{bmatrix} h_1-\lambda & h_0 & 0 & \cdots & 0 \\ h_2 & h_1-\lambda & h_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \cdots & h_0 \\ h_n & h_{n-1} & h_{n-2} & \cdots & h_1-\lambda \end{bmatrix}.$$

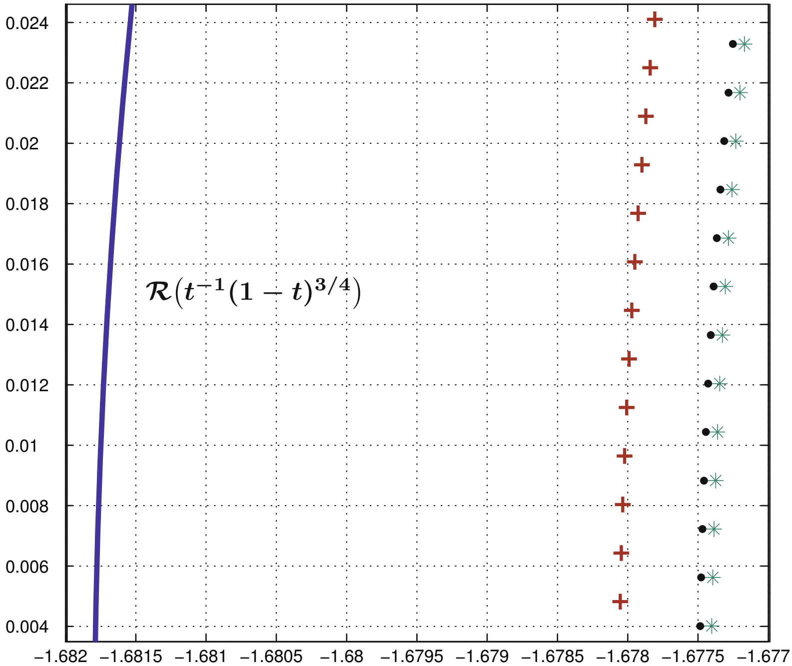

FIGURE 1. The picture shows a piece of $\mathcal{R}(a)$ for the symbol $a(t) = t^{-1}(1-t)^{3/4}$ (solid line) located "far" from zero. The dots are $\operatorname{sp} T_{4096}(a)$ calculated by *Matlab*. The crosses and the stars are the approximations obtained by using 2 and 3 terms of (1.5), respectively.

Applying Cramer's rule to (2.2) we obtain

$$\left[T_{n+1}^{-1}(h-\lambda t)\right]_{(n+1,1)} = (-1)^{n+2}\frac{D_n(a-\lambda)}{D_{n+1}(h-\lambda t)}. \tag{2.3}$$

We claim that $h(t)-\lambda t$ is invertible in H^∞. To see this, we must show that $h(t) \neq \lambda t$ for all $t \in \overline{\mathbb{D}}$ and each $\lambda \in \mathcal{D}(a)$. Let λ be a point in $\mathcal{D}(a)$. For each $t \in \mathbb{T}$ we have $h(t) \neq \lambda t$ because $\lambda \notin \partial\mathcal{D}(a) = \mathcal{R}(a)$. By assumption, $\operatorname{wind}_\lambda(a) = -1$ and thus

$$\begin{aligned} -1 = \operatorname{wind}_0(a-\lambda) &= \operatorname{wind}_0(t^{-1}h(t)-\lambda) = \operatorname{wind}_0\big(t^{-1}\big(h(t)-\lambda t\big)\big) \\ &= \operatorname{wind}_0(t^{-1}) + \operatorname{wind}_0(h(t)-\lambda t) = -1 + \operatorname{wind}_0(h(t)-\lambda t). \end{aligned}$$

It follows that $\operatorname{wind}_0(h(t)-\lambda t) = 0$, which means that the origin does not belong to the inside domain of the curve $\{h(t)-\lambda t \colon t \in \mathbb{T}\}$ (see [10, p. 204]). As $h \in H^\infty$, this shows that $h(t) \neq \lambda t$ for all $t \in \mathbb{D}$ and proves our claim.

If b is invertible in H^∞, then $T_{n+1}^{-1}(b) = T_{n+1}(1/b)$. Thus, the $(n+1,1)$ entry of the matrix $T_{n+1}^{-1}(h(t) - \lambda t)$ is in fact the nth Fourier coefficient of $(h(t) - \lambda t)^{-1}$,

$$\left[T_{n+1}^{-1}(h(t) - \lambda t)\right]_{(n+1,1)} = \left[\frac{1}{h(t) - \lambda t}\right]_n .$$

Inserting this in (2.3) we get

$$D_n(a - \lambda) = (-1)^{n+2} D_{n+1}(h(t) - \lambda t) \left[\frac{1}{h(t) - \lambda t}\right]_n = (-1)^n h_0^{n+1} \left[\frac{1}{h(t) - \lambda t}\right]_n ,$$

which completes the proof. □

Expression (2.1) says that the determinant $D_n(a - \lambda)$ can be expressed as the Fourier integral

$$D_n(a - \lambda) = (-1)^n h_0^{n+1} \int_{-\pi}^{\pi} \frac{e^{-in\theta}}{h(e^{i\theta}) - \lambda e^{i\theta}} \frac{d\theta}{2\pi},$$

which is our starting point to find an asymptotic expansion for the eigenvalues of $T_n(a)$. There are two major contributions to this integral. The first comes from λ, when it is close to $\mathcal{R}(a)$, and the second results from the singularity at the point 1. We will analyze them in separate sections.

3. Contribution of λ to the asymptotic behavior of D_n

Defining

$$b(z, \lambda) := \frac{1}{h(z) - \lambda z},$$

we have

$$b_n(\lambda) = \int_{-\pi}^{\pi} b(e^{i\theta}, \lambda) e^{-in\theta} \frac{d\theta}{2\pi}. \tag{3.1}$$

From (2.1) we conclude that

$$D_n(a - \lambda) = (-1)^n h_0^{n+1} b_n(\lambda). \tag{3.2}$$

Lemma 3.1. *Let $a(t) = t^{-1}h(t)$ be a symbol such that $\mathcal{R}(a)$ is a Jordan curve in $\mathbb{C}$. Let W_0 be a small open neighborhood of zero in $\mathbb{C}$. Assume that h has an analytic extension to an open neighborhood W of $\mathbb{T} \setminus \{1\}$ in $\mathbb{C}$ not containing the point 1. Then, for each $\lambda \in \mathcal{D}(a) \setminus W_0$ sufficiently close to $\mathcal{R}(a)$, there exists a unique point t_λ in $W \setminus \overline{\mathbb{D}}$ such that $a(t_\lambda) = \lambda$. Moreover, the point t_λ is a simple pole for b.*

Proof. Without loss of generality, we may assume that the extension of a to W is bounded. As $h \in H^\infty$, this extension must map $W \setminus \overline{\mathbb{D}}$ to $\mathcal{D}(a) \cap a(W)$. As the range of a has no loops, we have $a'(t) \neq 0$ for all $t \in \mathbb{T}$. Consider the compact set $S := \{t \in \mathbb{T} \colon a(t) \notin W_0\}$. For every $t \in S$, there exists an open neighborhood V_t of t in $\mathbb{C}$ with $V_t \subset W$ such that $a'(t) \neq 0$ for each $t \in V_t$. Thus, there is an open set U_t such that $t \in U_t \subset V_t$ and a is a conformal map (and hence bijective) from U_t to $a(U_t)$. As S is compact, we can take a finite sub-cover from $\{U_t\}_{t \in S}$, say $U := \cup_{i=1}^{M} U_{t_i}$. It follows that a is a conformal map (and hence bijective) from $U \supset S$ to $a(U) \supset a(S)$; see Figure 2. The lemma then holds for every $\lambda \in a(U) \cap (\mathcal{D}(a) \setminus W_0)$. Finally, since $a'(t_\lambda) \neq 0$, the point t_λ must be a simple pole of b. □

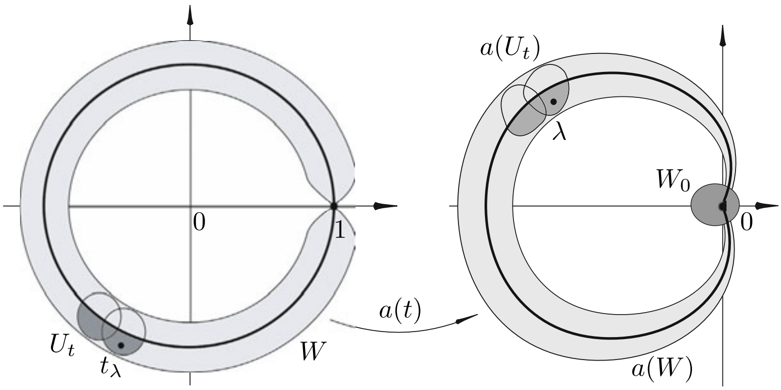

FIGURE 2. The map $a(t)$ over the unit circle.

Now using that t_λ is a simple pole of b, we split b as follows:

$$b(z,\lambda) = \frac{1}{z(a(z)-\lambda)} = \frac{1}{t_\lambda a'(t_\lambda)(z-t_\lambda)} + f_0(z,\lambda). \tag{3.3}$$

Here f_0 is analytic with respect to z in W and uniformly bounded with respect to λ in $a(W) \setminus W_0$. We calculate the Fourier coefficients of the first term in (3.3) directly and integrate the second term to get

$$b_n(\lambda) = \frac{-1}{t_\lambda^{n+2} a'(t_\lambda)} + \mathcal{I}, \tag{3.4}$$

where

$$\mathcal{I} := \int_{-\pi}^{\pi} f_0\big(e^{i\theta},\lambda\big) e^{-in\theta} \frac{d\theta}{2\pi}.$$

The first term in (3.4) times $(-1)^n h_0^{n+1}$ is the contribution of t_λ to the asymptotic expansion of $D_n(a-\lambda)$; see (3.2). The function f_0 has a singularity at $z = 1$ and we use this fact to expand $\mathcal{I}$ in the following section.

4. Contribution of 1 to the asymptotic behavior of D_n

In this section, we will show that the value of $\mathcal{I}$ in (3.4) depends mainly on the singularity at the point 1. Let us write $b(\theta,\lambda)$ and $f_0(\theta,\lambda)$ instead of $b\big(e^{i\theta},\lambda\big)$ and $f_0\big(e^{i\theta},\lambda\big)$, respectively. Let $\{\phi_1,\phi_2\}$ be a smooth partition of unity over the segment $[-\pi,\pi]$, which means that $\phi_1,\phi_2 \in C^\infty[-\pi,\pi]$, $\phi_1(\theta)+\phi_2(\theta) = 1$ for all

$\theta \in [-\pi, \pi]$, the support of ϕ_1 is contained in $[-\pi, -\varepsilon] \cup [\varepsilon, \pi]$, and the support of ϕ_2 is in $[-\delta, \delta]$, where $0 < \varepsilon < \delta$ are small constants. By pasting segments $[-\pi, \pi]$ in both directions, we can continue ϕ_1 and ϕ_2 to the entire real line $\mathbb{R}$, and we will think of these two functions in that way.

Lemma 4.1. *For every sufficiently small positive δ, we have*

$$\mathcal{I} = \int_{-\delta}^{\delta} \phi_2(\theta) b(\theta, \lambda) e^{-in\theta} \frac{d\theta}{2\pi} + Q_1(n, \lambda), \tag{4.1}$$

where $Q_1(n, \lambda) = \mathcal{O}(1/n^\infty)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$.

Proof. Using the partition of unity $\{\phi_1, \phi_2\}$, we write $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2$ where

$$\mathcal{I}_1 := \int_{\varepsilon}^{2\pi-\varepsilon} \phi_1(\theta) f_0(\theta, \lambda) e^{-in\theta} \frac{d\theta}{2\pi}, \quad \mathcal{I}_2 := \int_{-\delta}^{\delta} \phi_2(\theta) f_0(\theta, \lambda) e^{-in\theta} \frac{d\theta}{2\pi}.$$

The function $\phi_1(\theta) f_0(\theta, \lambda)$ belongs to $C^\infty[\varepsilon, 2\pi - \varepsilon]$. Thus by [14, p. 95], we obtain that $\mathcal{I}_1 = \mathcal{O}(1/n^\infty)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$.

Using (3.3) and writing $h(\theta)$ instead of $h\big(e^{i\theta}\big)$, we arrive at $\mathcal{I}_2 = \mathcal{I}_{21} + \mathcal{I}_{22}$ where

$$\mathcal{I}_{21} := \int_{-\delta}^{\delta} \frac{\phi_2(\theta) e^{-in\theta}}{h(\theta) - \lambda e^{i\theta}} \frac{d\theta}{2\pi}, \quad \mathcal{I}_{22} := \frac{-1}{t_\lambda a'(t_\lambda)} \int_{-\delta}^{\delta} \frac{\phi_2(\theta) e^{-in\theta}}{e^{i\theta} - t_\lambda} \frac{d\theta}{2\pi}. \tag{4.2}$$

Once more, the function $\phi_2(\theta) / \big(e^{i\theta} - t_\lambda\big)$ belongs to $C^\infty[-\delta, \delta]$, we thus conclude that $\mathcal{I}_{22} = \mathcal{O}(1/n^\infty)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$. ☐

Expression (4.1) says that the value of $\mathcal{I}$ basically depends on the integrand $b(\theta, \lambda) e^{-in\theta}$ at $\theta = 0$. As we can take δ as small as we desire, we can assume that θ is arbitrarily close to zero. Keeping this idea in mind, we will develop an asymptotic expansion for b. For future reference, we rewrite (4.1) as

$$\mathcal{I} = \mathcal{I}_{21} + Q_1(n, \lambda), \tag{4.3}$$

where $Q_1(n, \lambda) = \mathcal{O}(1/n^\infty)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$.

Lemma 4.2. *For every sufficiently small positive δ,*

$$\mathcal{I}_{21} = -\sum_{s=0}^{\infty} \frac{1}{\lambda^{s+1}} \int_{-\delta}^{\delta} \frac{\phi_2(\theta) h^s(\theta) e^{-in\theta}}{e^{i\theta(s+1)}} \frac{d\theta}{2\pi}. \tag{4.4}$$

Proof. From (4.2) we have

$$\mathcal{I}_{21} = \int_{-\delta}^{\delta} \phi_2(\theta) b(\theta, \lambda) e^{-in\theta} \frac{d\theta}{2\pi}. \tag{4.5}$$

Note that

$$b(\theta, \lambda) = \frac{1}{h(\theta) - \lambda e^{i\theta}} = \frac{-1}{\lambda e^{i\theta}} \cdot \frac{1}{1 - \lambda^{-1} e^{-i\theta} h(\theta)}.$$

As $|h(\theta)| \to 0$ when $\theta \to 0$, there exists a small positive constant δ such that

$$\big|\lambda^{-1} e^{-i\theta} h(\theta)\big| < 1$$

for every $|\theta| < \delta$. Thus,

$$b(\theta,\lambda) = \frac{-1}{\lambda e^{i\theta}} \sum_{s=0}^{\infty} \left(\lambda^{-1} e^{-i\theta} h(\theta)\right)^s = -\sum_{s=0}^{\infty} \frac{h^s(\theta)}{\lambda^{s+1} e^{i\theta(s+1)}} \tag{4.6}$$

for every $|\theta| < \delta$. Inserting (4.6) in (4.5) finishes the proof. □

We will use the notation

$$\mathcal{I}_{21s} := \frac{1}{\lambda^{s+1}} \int_{-\delta}^{\delta} \frac{\phi_2(\theta) h^s(\theta) e^{-in\theta}}{e^{i\theta(s+1)}} \frac{d\theta}{2\pi}.$$

Because $\phi_2(\theta)e^{-i\theta} \in C^\infty[-\delta,\delta]$, we have $\mathcal{I}_{21s}|_{s=0} = \mathcal{O}(1/n^\infty)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$. With the previous notation, we can rewrite (4.4) as

$$\mathcal{I}_{21} = -\sum_{s=1}^{\infty} \mathcal{I}_{21s} + Q_2(n,\lambda), \tag{4.7}$$

where $Q_2(n,\lambda) = \mathcal{O}(1/n^\infty)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$.

Finally we will work with $\mathcal{I}_{21s}$ and for this purpose we need the following well-known result, which is, for example, in [14, p. 97].

Theorem 4.3. *Let $\beta > 0$, $\delta > 0$, $v \in C^\infty[0,\delta]$, $v^{(s)}(\delta) = 0$ for all $s \geq 0$. Then, as $n \to \infty$,*

$$\int_0^\delta \theta^{\beta-1} v(\theta) e^{in\theta} d\theta \approx \sum_{s=0}^{\infty} \frac{a_s}{n^{s+\beta}},$$

where

$$a_s = \frac{v^{(s)}(0)}{s!} \Gamma(s+\beta) i^{s+\beta} \tag{4.8}$$

and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is Euler's Gamma function.

Lemma 4.4. *Let $h(t) = (1-t)^\alpha f(t)$ with $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$ and $f \in C^\infty(\mathbb{T})$. Then,*

$$\mathcal{I}_{21} = \frac{f(1)\Gamma(\alpha+1)\sin(\alpha\pi)}{\pi\lambda^2 n^{\alpha+1}} + R_1(n,\lambda), \tag{4.9}$$

where $R_1(n,\lambda) = \mathcal{O}(1/n^{\alpha+\alpha_0+1})$ with $\alpha_0 = \min\{\alpha, 1\}$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$.

Proof. It is easy to verify that $h(\theta) = (-i\theta)^\alpha v(\theta) f\left(e^{i\theta}\right)$, where the function v equals $\left(i\theta^{-1}(1-e^{i\theta})\right)^\alpha$, the branch of the αth power being the one corresponding to the argument in $(-\pi,\pi]$; note that for every sufficiently small positive δ we have $v \in C^\infty[-\delta,\delta]$ and $v(0) = 1$. Thus,

$$\begin{aligned}\mathcal{I}_{21s} &= \frac{1}{\lambda^{s+1}} \int_{-\delta}^{\delta} \phi_2(\theta) h^s(\theta) e^{-i\theta(n+s+1)} \frac{d\theta}{2\pi} \\ &= \frac{(-i)^{\alpha s}}{\lambda^{s+1}} \int_{-\delta}^{\delta} \phi_2(\theta) \theta^{\alpha s} v^s(\theta) f^s\left(e^{i\theta}\right) e^{-i\theta(n+s+1)} \frac{d\theta}{2\pi}.\end{aligned}$$

The last integral can be written as

$$\begin{aligned}\mathcal{I}_{21s} &= \int_{-\delta}^{\delta} \theta^{\beta-1} w(\theta) e^{-in\theta} d\theta \\ &= \int_{-\delta}^{0} \theta^{\beta-1} w(\theta) e^{-in\theta} d\theta + \int_{0}^{\delta} \theta^{\beta-1} w(\theta) e^{-in\theta} d\theta \\ &= \int_{0}^{\delta} (-\tau)^{\beta-1} w(-\tau) e^{in\tau} d\tau + \int_{0}^{\delta} \theta^{\beta-1} w(\theta) e^{-in\theta} d\theta = \mathcal{I}_{21s1} + \mathcal{I}_{21s2}, \quad (4.10)\end{aligned}$$

where $\beta := \alpha s + 1$, $w(\theta) := (-i)^{\alpha s} \phi_2(\theta) v^s(\theta) f^s\left(e^{i\theta}\right) e^{-i\theta(s+1)}/(2\pi\lambda^{s+1})$, and

$$\mathcal{I}_{21s1} := (-1)^{\beta-1} \int_{0}^{\delta} \theta^{\beta-1} w(-\theta) e^{in\theta} d\theta, \quad \mathcal{I}_{21s2} := \int_{0}^{\delta} \theta^{\beta-1} w(\theta) e^{-in\theta} d\theta.$$

Note that $w(\pm\theta) \in C^{\infty}[0,\delta]$ and $w^{(s)}(\pm\delta) = 0$ for all $s \in \mathbb{N}$ because $\phi_2(\theta) \equiv 0$ in a small neighborhood of $\pm\delta$. Applying (4.8) to $\mathcal{I}_{21s1}$ and $\overline{\mathcal{I}_{21s2}}$, we obtain

$$\mathcal{I}_{21s1} = \frac{(-1)^{\alpha s} w(0) \Gamma(\alpha s + 1) i^{\alpha s + 1}}{n^{\alpha s + 1}} + Q_3(s,n,\lambda)$$

and

$$\mathcal{I}_{21s2} = \frac{w(0) \Gamma(\alpha s + 1) i^{-\alpha s - 1}}{n^{\alpha s + 1}} + Q_4(s,n,\lambda), \quad (4.11)$$

where $Q_3(s,n,\lambda)$ and $Q_4(s,n,\lambda)$ are $\mathcal{O}(1/n^{\alpha s+2})$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$. Substitution of (4.11) in (4.10) yields

$$\begin{aligned}\mathcal{I}_{21s} =& \frac{w(0)\Gamma(\alpha s + 1)}{n^{\alpha s+1}} \left(i^{-\alpha s - 1} + (-1)^{\alpha s} i^{\alpha s + 1}\right) + Q_5(s,n,\lambda) \\ =& \frac{-C_s}{\lambda^{s+1} n^{\alpha s + 1}} + Q_5(s,n,\lambda) \quad (4.12)\end{aligned}$$

where

$$C_s := \frac{f^s(1)\Gamma(\alpha s + 1)\sin(\alpha \pi s)}{\pi} \quad (4.13)$$

and $Q_5(s,n,\lambda) = \mathcal{O}(1/n^{\alpha s+2})$ as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$. From (4.7) and (4.12) we obtain

$$\mathcal{I}_{21} = \frac{C_1}{\lambda^2 n^{\alpha+1}} + R_1(n,\lambda),$$

where $R_1(n,\lambda) = \mathcal{O}(1/n^{\alpha+\alpha_0+1})$ as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$. Here $\alpha_0 := \min\{\alpha, 1\}$. □

The previous calculation gives us the main asymptotic term for $\mathcal{I}_{21}$. If more terms are needed, say m, we must expand $\mathcal{I}_{21}$ from $\mathcal{I}_{21s}|_{s=1}$ to $\mathcal{I}_{21s}|_{s=m}$ and expand each $\mathcal{I}_{21s}$ to m terms, after which, according to the value of α, we need to select the first m principal terms.

Finally we put all the lemmas together to prove Theorem 1.1.

Proof of Theorem 1.1. The proof of this theorem is a direct application of equations (3.2), (3.4), (4.3) and (4.9). □

5. Individual eigenvalues

In order to find the eigenvalues of the matrices $T_n(a)$, we need to solve the equations $D_n(a-\lambda)=0$. We start this section by locating the zeros of $D_n(a-\lambda)$.

Let W_0 be a small open neighborhood of zero in $\mathbb{C}$ and $\omega_n := \exp(-2\pi i/n)$. For each n there exist integers n_1 and n_2 such that $\omega_n^{n_1}, \omega_n^{n-n_2} \in a^{-1}(W_0)$ but $\omega_n^{n_1+1}, \omega_n^{n-n_2-1} \notin a^{-1}(W_0)$. Recall that $\lambda = a(t_\lambda)$. Take an integer j satisfying $n_1 < j < n-n_2$. Using the relations

$$\frac{1}{t_\lambda^2 a'(t_\lambda)} = \frac{1}{\omega_j^2 a'(\omega_n^j)} + \mathcal{O}\left(\left|t_\lambda - \omega_n^j\right|\right)$$

and

$$\frac{1}{a^2(t_\lambda)} = \frac{1}{a^2(\omega_n^j)} + \mathcal{O}\left(\left|t_\lambda - \omega_n^j\right|\right),$$

where t_λ belongs to a small neighborhood of ω_n^j, we see that the determinant $D_n(a-\lambda)$ in (1.3) equals

$$\begin{aligned}
&(-h_0)^{n+1}\left(\mathcal{T}_1 - \mathcal{T}_2 + \frac{1}{t_\lambda^n}\mathcal{O}\left(\left|t_\lambda - \omega_n^j\right|\right) + \frac{1}{n^{\alpha+1}}\mathcal{O}\left(\left|t_\lambda - \omega_n^j\right|\right) + Q_6(n,t_\lambda)\right)\\
&= (-h_0)^{n+1}\left(\mathcal{T}_1 - \mathcal{T}_2 + \mathcal{O}\left(\left|\frac{t_\lambda - \omega_n^j}{t_\lambda^n}\right|\right) + \mathcal{O}\left(\frac{\left|t_\lambda - \omega_n^j\right|}{n^{\alpha+1}}\right) + Q_6(n,t_\lambda)\right), \quad (5.1)
\end{aligned}$$

where $Q_6(n,t_\lambda) = \mathcal{O}(1/n^{\alpha+\alpha_0+1})$ as $n\to\infty$, uniformly with respect to t_λ in $W \setminus a^{-1}(W_0)$, and where t_λ belongs to a small neighborhood of ω_n^j. Here

$$\mathcal{T}_1 := \frac{1}{t_\lambda^n \omega_n^{2j} a'(\omega_n^j)}, \quad \mathcal{T}_2 := \frac{C_1}{a^2(\omega_n^j) n^{\alpha+1}},$$

and $\alpha_0 := \min\{\alpha, 1\}$. Recall C_1 from (4.13). Expression (5.1) makes sense only when t_λ is sufficiently "close" to ω_n^j and thus, it is necessary to know whether there exists a zero of $D_n(a-\lambda)$ "close" to ω_n^j. Let

$$t_\lambda = (1+\rho)\exp(i\theta).$$

It is easy to verify that $\mathcal{T}_1 - \mathcal{T}_2 = 0$ if and only if

$$\rho = \left(\frac{|a(\omega_n^j)|^2 n^{\alpha+1}}{|C_1 a'(\omega_n^j)|}\right)^{1/n} - 1 \tag{5.2}$$

and

$$\theta = \theta_j = \frac{1}{n}\arg\left(\frac{a^2(\omega_n^j)}{C_1 \omega_n^{2j} a'(\omega_n^j)}\right) - \frac{2\pi j}{n}$$

for some $j \in \{0,\ldots,n-1\}$. When n tends to infinity, (5.2) shows that ρ remains positive and $\rho \to 0$. The function $\mathcal{T}_1 - \mathcal{T}_2$ has n zeros with respect to $\lambda \in \mathcal{D}(a)$ given by

$$a\left((1+\rho)e^{i\theta_0}\right), \quad \ldots, \quad a\left((1+\rho)e^{i\theta_{n-1}}\right).$$

As Lemma 3.1 establishes a 1–1 correspondence between λ and t_λ, the function $D_n(a-\lambda)$ is analytic with respect to λ in $a(W)\setminus W_0$, that is, analytic with respect to t_λ in $W \setminus a^{-1}(W_0)$. We can therefore suppose that $\mathcal{T}_1 - \mathcal{T}_2$ has n zeros with respect to t_λ in the exterior of $\overline{\mathbb{D}}$ given by

$$t_0 := (1+\rho)e^{i\theta_0}, \quad \ldots, \quad t_{n-1} := (1+\rho)e^{i\theta_{n-1}}.$$

We take the function "arg" in the interval $(-\pi,\pi]$. Thus, $t_j = (1+\rho)e^{i\theta_j}$ is the nearest zero to ω_n^j. Consider the neighborhood E_j of t_j sketched in Figure 3.

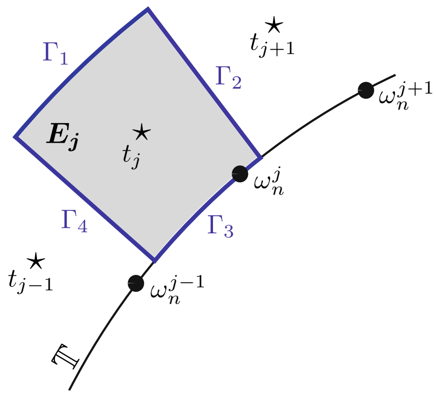

FIGURE 3. The neighborhood E_j of t_j in the complex plane.

The boundary of E_j is $\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$. We have chosen radial segments Γ_2 and Γ_4 so that their length is $1/n^\epsilon$ with $\epsilon \in (0,\alpha_0)$ and all the points in Γ_2 have the common argument $(\theta_{j+1}+\theta_j)/2$, while all the points in Γ_4 have the common argument $(\theta_{j-1}+\theta_j)/2$. As we can see in Figure 3, these points run from the unit circle $\mathbb{T}$ to $(1+1/n^\epsilon)\mathbb{T}$. Note also that $\Gamma_1 \subset (1+1/n^\epsilon)\mathbb{T}$ and $\Gamma_3 \subset \mathbb{T}$.

Theorem 5.1. *Suppose* $a(t) = t^{-1}h(t)$ *is a symbol with properties* 1 *to* 4. *Let* $\epsilon \in (0,\alpha_0)$ *be a constant. Then there exists a family of sets* $\{E_j\}_{j=n_1+1}^{n-n_2-1}$ *in* $\mathbb{C}$ *such that*

1. $\{E_j\}_{j=n_1+1}^{n-n_2-1}$ *is a family of pairwise disjoint open sets,*
2. $\operatorname{diam}(E_j) \le 2/n^\epsilon$,
3. $\omega_n^j \in \partial E_j$,
4. $D_n\big(a - a(t_\lambda)\big) = D_n(a-\lambda)$ *has exactly one zero in each* E_j.

Here $\alpha_0 := \min\{\alpha, 1\}$ *and* $\operatorname{diam}(E_j) := \sup\{|z_1 - z_2| \colon z_1, z_2 \in E_j\}$.

Proof. Assertions 1, 2, and 3 can be deduced from the above construction. We prove assertion 4 by studying the behavior of $|D_n(a-\lambda)|$ in dependence on $t_\lambda \in \Gamma$.

For $t_\lambda \in \Gamma_1$ we have, as $n \to \infty$,

$$\begin{aligned}
|\mathcal{T}_1|_{\Gamma_1} &= \frac{1}{|a'(\omega_n^j)|} \cdot \left(1 + \frac{1}{n^\epsilon}\right)^{-n} = \frac{\exp(-n^{1-\epsilon})}{|a'(\omega_n^j)|} + \mathcal{O}\left(\frac{\exp(-n^{1-\epsilon})}{n^{2\epsilon-1}}\right), \\
|\mathcal{T}_2|_{\Gamma_1} &= \frac{1}{n^{\alpha+1}} \cdot \left|\frac{C_1}{a^2(\omega_n^j)}\right|, \\
\left|\mathcal{O}\left(\left|\frac{t_\lambda - \omega_n^j}{t_\lambda^n}\right|\right)\right|_{\Gamma_1} &= \mathcal{O}\left(\frac{\exp(-n^{1-\epsilon})}{n^\epsilon}\right), \\
\left|\mathcal{O}\left(\frac{|t_\lambda - \omega_n^j|}{n^{\alpha+1}}\right)\right|_{\Gamma_1} &= \mathcal{O}\left(\frac{1}{n^{\alpha+\epsilon+1}}\right),
\end{aligned}$$

and $|Q_6(n, t_\lambda)|_{\Gamma_1} = \mathcal{O}(1/n^{\alpha+\alpha_0+1})$. When n goes to infinity, the absolute value of $\mathcal{T}_2$ decreases at polynomial speed over Γ_1, while the absolute values of the remaining terms in (5.1) are smaller over Γ_1. Thus,

$$\left|\frac{D_n(a-\lambda)}{h_0^{n+1}}\right|_{\Gamma_1} = \frac{1}{n^{\alpha+1}} \cdot \left|\frac{C_1}{a^2(\omega_n^j)}\right| + \mathcal{O}\left(\frac{1}{n^{\alpha+\epsilon+1}}\right) \text{ as } n \to \infty.$$

For $t_\lambda \in \Gamma_3$ we get, as $n \to \infty$,

$$\begin{aligned}
|\mathcal{T}_1|_{\Gamma_3} &= \frac{1}{|a'(\omega_n^j)|}, \quad |\mathcal{T}_2|_{\Gamma_3} = \frac{1}{n^{\alpha+1}} \cdot \left|\frac{C_1}{a^2(\omega_n^j)}\right|, \\
\left|\mathcal{O}\left(\left|\frac{t_\lambda - \omega_n^j}{t_\lambda^n}\right|\right)\right|_{\Gamma_3} &= \mathcal{O}\left(\frac{1}{n}\right), \\
\left|\mathcal{O}\left(\frac{|t_\lambda - \omega_n^j|}{n^{\alpha+1}}\right)\right|_{\Gamma_3} &= \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right),
\end{aligned}$$

and $|Q_6(n, t_\lambda)|_{\Gamma_3} = \mathcal{O}(1/n^{\alpha+\alpha_0+1})$. When n goes to infinity, the modulus of $\mathcal{T}_1$ remains constant over Γ_3, while the moduli of the remaining terms in (5.1) are smaller there. Consequently,

$$\left|\frac{D_n(a-\lambda)}{h_0^{n+1}}\right|_{\Gamma_3} = \frac{1}{|a'(\omega_n^j)|} + \mathcal{O}\left(\frac{1}{n}\right) \text{ as } n \to \infty.$$

As for the radial segments Γ_2 and Γ_4, we start by showing that $\mathcal{T}_1$ and $-\mathcal{T}_2$ have the same argument there. Since t_j is a zero of $\mathcal{T}_1 - \mathcal{T}_2$, we deduce that

$$\arg\left(\frac{1}{t_j^n \omega_n^{2j} a'(\omega_n^j)}\right) = \arg\left(\frac{C_1}{a^2(\omega_n^j) n^{\alpha+1}}\right)$$

and thus

$$-n\theta_j + \arg\left(\frac{1}{\omega_n^{2j} a'(\omega_n^j)}\right) = \arg\left(\frac{C_1}{a^2(\omega_n^j)}\right). \tag{5.3}$$

For $t_\lambda \in \Gamma_2$ we have

$$\begin{aligned}\arg(\mathcal{T}_1) &= \arg\left(\frac{1}{t_\lambda^n \omega_n^{2j} a'(\omega_n^j)}\right)\\ &= -\frac{n}{2}(\theta_{j-1}+\theta_j) + \arg\left(\frac{1}{\omega_n^{2j} a'(\omega_n^j)}\right) = \frac{n}{2}(\theta_j - \theta_{j-1}) + \arg\left(\frac{C_1}{a^2(\omega_n^j)}\right)\\ &= \pi + \arg\left(\frac{C_1}{a^2(\omega_n^j)}\right) = \arg(-\mathcal{T}_2).\end{aligned}$$

Here, the third line is due to (5.3). In addition, as $n \to \infty$,

$$\left|\mathcal{O}\left(\left|\frac{t_\lambda - \omega_n^j}{t_\lambda^n}\right|\right)\right|_{\Gamma_2} = \mathcal{O}\left(\frac{1}{n^\epsilon |t_\lambda|^n}\right), \quad \left|\mathcal{O}\left(\frac{|t_\lambda - \omega_n^j|}{n^{\alpha+1}}\right)\right|_{\Gamma_2} = \mathcal{O}\left(\frac{1}{n^{\alpha+\epsilon+1}}\right),$$

and $|Q_6(n,t_\lambda)|_{\Gamma_2} = \mathcal{O}(1/n^{\alpha+\alpha_0+1})$. Furthermore,

$$\begin{aligned}\left|\frac{D_n(a-\lambda)}{h_0^{n+1}}\right|_{\Gamma_2} &= \frac{1}{|t_\lambda^n a'(\omega_n^j)|} + \mathcal{O}\left(\frac{1}{n^\epsilon |t_\lambda|^n}\right)\\ &\qquad + \frac{1}{n^{\alpha+1}} \cdot \left|\frac{C_1}{a^2(\omega_n^j)}\right| + \mathcal{O}\left(\frac{1}{n^{\alpha+\epsilon+1}}\right)\end{aligned}$$

over Γ_2 when $n \to \infty$. The situation is similar for the segment Γ_4.

From the previous analysis of $|D_n(a-\lambda)|$ over Γ we infer that for every sufficiently large n we have

$$|\mathcal{T}_1 - \mathcal{T}_2|_\Gamma \geq \frac{1}{2n^{\alpha+1}}\left|\frac{C_1}{a^2(\omega_n^j)}\right|$$

and

$$\left|\mathcal{O}\left(\left|\frac{t_\lambda - \omega_n^j}{t_\lambda^n}\right|\right) + \mathcal{O}\left(\frac{|t_\lambda - \omega_n^j|}{n^{\alpha+1}}\right) + Q_6(n,t_\lambda)\right|_\Gamma \leq \mathcal{O}\left(\frac{1}{n^{\alpha+\epsilon+1}}\right).$$

Hence by Rouché's theorem, $D_n(a-\lambda)/(-h_0)^{n+1}$ and $\mathcal{T}_1 - \mathcal{T}_2$ have the same number of zeros in E_j, that is, a unique zero. □

As a consequence of Theorem 5.1, we can iterate the variable t_λ in the equation $D_n(a-\lambda) = 0$, where $D_n(a-\lambda)$ is given by (1.3). In this fashion we find the unique eigenvalue of $T_n(a)$ which is located "close" to each ω_n^j. We thus rewrite the equation $D_n(a-\lambda) = 0$ in a small neighborhood of ω_n^j as

$$t_{\lambda_{j,n}} = n^{(\alpha+1)/n}\omega_n^j \left(\frac{a^2(t_{\lambda_{j,n}})}{C_1 a'(t_{\lambda_{j,n}}) t_{\lambda_{j,n}}^2}\right)^{\frac{1}{n}} \cdot \big(1 + Q_7(n,j)\big)^{-\frac{1}{n}}; \qquad (5.4)$$

recall C_1 from (4.13). Here the function $z^{1/n}$ takes its principal branch, specified by the argument in $(-\pi, \pi]$. Also notice that $Q_7(n,j) = \mathcal{O}(1/n^{\alpha_0})$ as $n \to \infty$, uniformly in $j \in (n_1, n - n_2)$, with n_1, n_2 as in Theorem 5.1.

Proof of Theorem 1.2. Equation (5.4) is an implicit expression for $t_{\lambda_{j,n}}$. We manipulate it to obtain two asymptotic terms for $t_{\lambda_{j,n}}$. Remember that λ belongs to $\mathcal{D}(a) \setminus W_0$; see Figure 2. We can choose W so thin that $\lambda_{j,n} = a(t_{\lambda_{j,n}})$, $a'(t_{\lambda_{j,n}})$, and $t_{\lambda_{j,n}}$ are bounded and not too close to zero. After expanding and multiplying the terms in brackets in (5.4), we obtain

$$t_{\lambda_{j,n}} = n^{(\alpha+1)/n}\omega_n^j \left(1 + \frac{1}{n}\log\left(\frac{a^2(t_{\lambda_{j,n}})}{C_1 a'(t_{\lambda_{j,n}})t_{\lambda_{j,n}}^2}\right) + Q_8(n,j)\right), \tag{5.5}$$

where $Q_8(n,j) = \mathcal{O}(1/n^{\alpha_0+1})$ as $n \to \infty$, uniformly with respect to j in $(n_1, n - n_2)$. Our first approximation for $t_{\lambda_{j,n}}$ is

$$t_{\lambda_{j,n}} = n^{(\alpha+1)/n}\omega_n^j\big(1 + Q_9(n,j)\big),$$

where $Q_9(n,j) = \mathcal{O}(1/n)$ as $n \to \infty$, uniformly in j from $(n_1, n-n_2)$. Replacing $t_{\lambda_{j,n}}$ by this approximation in (5.5) shows that $t_{\lambda_{j,n}}$ equals $n^{(\alpha+1)/n}\omega_n^j$ times

$$1 + \frac{1}{n}\log\left(\frac{a^2\big(n^{(\alpha+1)/n}\omega_n^j\big[1+Q_9(n,j)\big]\big)}{C_1 a'\big(n^{(\alpha+1)/n}\omega_n^j\big[1+Q_9(n,j)\big]\big)\big(n^{(\alpha+1)/n}\omega_n^j\big[1+Q_9(n,j)\big]\big)^2}\right),$$

plus $Q_{10}(n,j)$, where $Q_{10}(n,j) = \mathcal{O}(1/n^{\alpha_0+1})$ as $n \to \infty$, uniformly with respect to j in $(n_1, n-n_2)$. Now we use the analyticity of a and a' in W to obtain that $t_{\lambda_{j,n}}$ is $n^{(\alpha+1)/n}\omega_n^j$ times

$$1 + \frac{1}{n}\log\left(\frac{a^2\big(n^{(\alpha+1)/n}\omega_n^j\big)}{C_1 a'\big(n^{(\alpha+1)/n}\omega_n^j\big)\big(n^{(\alpha+1)/n}\omega_n^j\big)^2}\right) + Q_{11}(n,j),$$

where $Q_{11}(n,j) = \mathcal{O}(1/n^{\alpha_0+1})$ as $n \to \infty$, uniformly in $j \in (n_1, n-n_2)$. Taking into account that

$$\frac{a^2\big(n^{(\alpha+1)/n}\omega_n^j\big)}{C_1 a'\big(n^{(\alpha+1)/n}\omega_n^j\big)\big(n^{(\alpha+1)/n}\omega_n^j\big)^2} = \frac{a^2(\omega_n^j)}{C_1 a'(\omega_n^j)\omega_n^{2j}} + \mathcal{O}\left(\frac{\log n}{n}\right) \text{ as } n \to \infty,$$

we can simplify the expression for $t_{\lambda_{j,n}}$ to

$$t_{\lambda_{j,n}} = n^{(\alpha+1)/n}\omega_n^j\left(1 + \frac{1}{n}\log\left(\frac{a^2(\omega_n^j)}{C_1 a'(\omega_n^j)\omega_n^{2j}}\right) + R_2(n,j)\right),$$

where $R_2(n,j) = \mathcal{O}(1/n^{\alpha_0+1}) + \mathcal{O}(\log n/n^2)$ as $n \to \infty$, uniformly with respect to j in $(n_1, n-n_2)$. $\square$

Proof of Theorem 1.3. Note that

$$n^{(\alpha+1)/n} = \exp\left((\alpha+1)\frac{\log n}{n}\right) = 1 + (\alpha+1)\frac{\log n}{n} + \mathcal{O}\left(\frac{\log n}{n}\right)^2 \text{ as } n \to \infty. \tag{5.6}$$

Inserting (5.6) in (1.4) we obtain

$$t_{\lambda_{j,n}} = \omega_n^j\left(1 + (\alpha+1)\frac{\log n}{n} + \frac{1}{n}\log\left(\frac{a^2(\omega_n^j)}{C_1 a'(\omega_n^j)\omega_n^{2j}}\right) + Q_{12}(n,j)\right), \tag{5.7}$$

where $Q_{12}(n,j) = \mathcal{O}(1/n^{\alpha_0+1}) + \mathcal{O}(\log^2 n/n^2)$ as $n \to \infty$, uniformly with respect to $j \in (n_1, n-n_2)$. Applying the symbol a to (5.7), we see that, as $n \to \infty$,

$$\lambda_{j,n} = a\big(\omega_n^j\big) + (\alpha+1)\omega_n^j a'(\omega_n^j)\frac{\log n}{n} + \frac{\omega_n^j a'(\omega_n^j)}{n}\log\left(\frac{a^2(\omega_n^j)}{C_1 a'(\omega_n^j)\omega_n^{2j}}\right) + a'(\omega_n^j)Q_{12}(n,j) + \mathcal{O}\left(\frac{\log^2 n}{n^2}\right). \qquad \square$$

6. An example

The symbol studied by Dai, Geary, and Kadanoff [11] is

$$a(t) = \left(2 - t - \frac{1}{t}\right)^{\gamma}(-t)^{\beta} = (-1)^{3\gamma+\beta}t^{\beta-\gamma}(1-t)^{2\gamma},$$

where $0 < \gamma < -\beta < 1$. In the case $\beta = \gamma - 1$, this function a becomes our symbol with $h(t) = (-1)^{4\gamma-1}(1-t)^{2\gamma}$. We omit the constant $(-1)^{4\gamma-1}$, because it is just a rotation. The conjecture of [11] is that $t_{\lambda_{j,n}} \approx n^{(2\gamma+1)/n}\exp(-2\pi ij/n)$. Expansions (1.4) and (1.5) prove this result, giving us an error bound and a mathematical justification.

Our results are valid outside a small open neighborhood W_0 of the origin. Let $W_0 = B_{1/5}(0)$ be the disk of radius 1/5 centered at zero. Table 1 shows the data of numerical computations. It reveals that the maximum error of (1.4) with one term is reduced by nearly 10 times when considering the second term; see also Figure 1.

n	256	512	1024	2048	4096
(1.4) with 1 term	1.6×10^{-2}	8.1×10^{-3}	4.1×10^{-3}	2.1×10^{-3}	1.0×10^{-3}
(1.4) with 2 terms	1.7×10^{-3}	4.5×10^{-4}	1.2×10^{-4}	3.2×10^{-5}	8.7×10^{-6}
(1.5) with 1 term	5.1×10^{-2}	2.8×10^{-2}	1.5×10^{-2}	8.3×10^{-3}	4.4×10^{-3}
(1.5) with 2 terms	1.5×10^{-2}	7.9×10^{-3}	4.1×10^{-3}	2.1×10^{-3}	1.0×10^{-3}
(1.5) with 3 terms	1.4×10^{-3}	4.3×10^{-4}	1.3×10^{-4}	3.7×10^{-5}	1.1×10^{-5}

Table 1. The table shows the maximum error obtained with our different formulas for the eigenvalues of $T_n\big(t^{-1}(1-t)^{3/4}\big)$ for different values of n. The data was obtained by comparison with the solutions given by *Matlab*, taking into account only the eigenvalues with absolute value greater than or equal to 1/5.

We also performed calculations with our expansions inside $W_0 = B_{1/5}(0)$, and although the error is nearly 8 times the one of outside, the approximation is still valid there because the distance between two consecutive eigenvalues is bigger than the one between an eigenvalue and the respective approximation given by (1.4) with two terms; compare Tables 1 and 2 and see Figure 4. Clearly, to describe

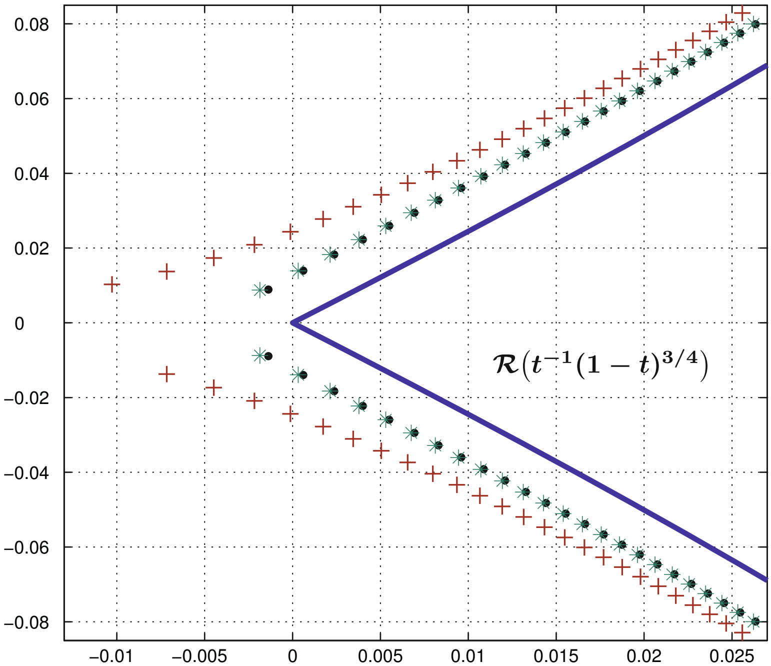

FIGURE 4. The picture shows a piece of $\mathcal{R}(a)$ for the symbol $a(t) = t^{-1}(1-t)^{3/4}$ (solid line), located "close" to zero. The dots are $\operatorname{sp} T_{4096}(a)$ calculated by *Matlab*. The crosses and the stars are the approximations obtained by using 2 and 3 terms of (1.5), respectively.

the asymptotic behavior of the eigenvalues of $T_n(a)$ completely with mathematical rigor, we need an expression valid inside W_0. We hope to do this in future work.

n	256	512	1024	2048	4096
(1.4) with 1 term	2.7×10^{-2}	2.1×10^{-2}	1.5×10^{-2}	1.1×10^{-2}	7.2×10^{-3}
(1.4) with 2 terms	6.4×10^{-3}	3.9×10^{-3}	2.3×10^{-3}	1.4×10^{-3}	8.2×10^{-4}
(1.5) with 1 term	3.9×10^{-2}	2.4×10^{-2}	1.4×10^{-2}	8.4×10^{-3}	5.0×10^{-3}
(1.5) with 2 terms	3.3×10^{-2}	2.5×10^{-2}	1.8×10^{-2}	1.2×10^{-2}	8.5×10^{-3}
(1.5) with 3 terms	2.7×10^{-3}	1.7×10^{-3}	1.1×10^{-3}	6.6×10^{-4}	3.9×10^{-4}

TABLE 2. The same as in Table 1, only now considering eigenvalues with absolute value less than 1/5.

We remark that if λ is an eigenvalue of $T_n(a)$ and $b_j(\lambda)$ is defined by (3.1), then $\big(b_j(\lambda)\big)_{j=0}^{n-1}$ is an eigenvector for λ provided $b_{n-1}(\lambda) \neq 0$. In a forthcoming paper we will employ this observation to study the asymptotics of the eigenvectors.

We finally want to emphasize that the results of this paper can be easily translated to the case where the symbol is $\overline{a(t)} = t(1-t^{-1})^{\alpha}\overline{f(t)}$.

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Real Berezin Transform and Asymptotic Expansion for Symmetric Spaces of Compact and Non-compact Type

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Abstract. We obtain formulas for the asymptotic expansion of the Berezin transform on symmetric spaces in terms of invariant differential operators associated with the Peter-Weyl decomposition under the maximal compact subgroup. A unified treatment makes it possible to derive the formulas for the complex (hermitian) as well as for the real case, and for all types of symmetric spaces (non-compact, compact and flat).

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1. Introduction

The Berezin transform is of central importance in the theory of deformation quantization of complex Kähler manifolds, in particular for the special case of symmetric spaces of hermitian type. In this case the eigenvalues of the Berezin transform are explicitly known, both in the non-compact case of hermitian bounded symmetric domains [UU] and for the compact hermitian symmetric spaces [Z2] arising from duality. Besides the spectral analysis, another important problem is the expansion of the Berezin transform into an asymptotic series of differential operators, as the deformation parameter ("inverse Planck constant") tends to infinity. More precisely, the well-known Toeplitz star- (or Moyal) products have asymptotic expansions which are closely related to that of the (inverse) Berezin transform [EU1], [EU2].

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For non-compact symmetric domains, the asymptotic expansion of the Berezin transform was obtained by Arazy and Ørsted [AO]. In a separate paper [EU1] we generalized this result to the case of *real* bounded symmetric domains, where again there is a natural "Berezin" transform which is closely related to the well-known Segal-Bargmann transformations. The dual situation of compact symmetric spaces (complex or real) was not considered in [AO] or [EU1], [EU2].

The purpose of this paper is to give the asymptotic expansion of the Berezin transform for symmetric spaces of compact and non-compact type, both in the classical complex setting of hermitian symmetric spaces and in the real setting for the Segal-Bargmann type Berezin transform. We present a uniform proof for all cases, showing that the compact type behaves quite similar to the dual non-compact situation. As usual the "flat" case, where the semi-simple covariance group degenerates into a semi-direct product, is included in the computations.

In a separate paper [EU2] we apply these results to obtain asymptotic expansions (in the Peter-Weyl context) for star products in the compact and non-compact situation, as well as for the so-called "star restrictions" which are their real counterparts.

Precursors of the results included here appeared in one of the talks by the first author at the IWOTA09 conference in Guanajuato, Mexico, in September 2009. Both authors thank the organizers for the invitation (which the second author was not able to accept in the end).

2. Berezin transform for hermitian symmetric spaces

It is well known that most Riemannian symmetric spaces, including all the classical ones, have a uniform description in terms of Jordan algebras and Jordan triples. We refer to [EU1], [L] for a detailed discussion of the Jordan theoretic background and notation. For any hermitian Jordan triple Z the (spectral) open unit ball Z^- is a hermitian bounded symmetric domain whose compact dual Z^+ is a Jordan theoretic analogue of the Grassmann manifold, containing Z as a Zariski open subset, i.e.,

$$Z^- \subset Z \subset Z^+.$$

All hermitian symmetric spaces (compact or non-compact) arise this way (the non-hermitian case will be studied in Section 3).

Note that the common "base point" is the origin 0; therefore the "non-compact" symmetric space Z^- is given in the "bounded" realization (unit ball) instead of the "unbounded" realization as a tube domain or Siegel domain.

We define real Lie groups

$$\begin{aligned} G^- &:= \mathrm{Aut}\,(Z^-) = \{\text{biholomorphic automorphisms of } Z^-\}, \\ G^+ &= \{\text{biholomorphic isometries of } Z^+\}, \\ G^0 &= Z \times K \quad (\text{semi-direct product}), \end{aligned}$$

where $K = \text{Aut}\,(Z) = \{g \in G^{\pm} :\ g(0) = 0\}$ is the Jordan triple automorphism group.

The three types of hermitian manifolds $Z^\bullet = Z^+, Z, Z^-$ give rise to reproducing kernel Hilbert spaces $H^2_\nu(Z^\bullet)$ of holomorphic functions, which play the role of quantization state spaces. Here ν is a deformation parameter ("inverse Planck constant"). Let $\mathcal{K}^\bullet(z,w)$ denote the reproducing kernel. The *Berezin transform* (related to the so-called Toeplitz-Berezin quantization calculus [BMS]) is a $G^\bullet$-invariant densely defined self-adjoint operator

$$\mathcal{B}^\bullet : L^2(Z^\bullet) \to L^2(Z^\bullet)$$

with integral kernel representation

$$(\mathcal{B}^\bullet f)(z) = \int\limits_{Z^\bullet} d\mu^\bullet(w)\ \frac{\mathcal{K}^\bullet(z,w)\,\mathcal{K}^\bullet(w,z)}{\mathcal{K}^\bullet(z,z)\,\mathcal{K}^\bullet(w,w)}\ f(w)$$

with respect to a suitably normalized $G^\bullet$-invariant measure $d\mu^\bullet$ specified below.

We will now discuss the three types of hermitian symmetric spaces separately. For $Z^\bullet = Z$, the *flat* case, $\nu > 0$ is arbitrary and $H^2_\nu(Z)$ is the Fock space of all entire functions $\psi \in \mathcal{O}(Z)$ satisfying

$$\|\psi\|^2_\nu = \int\limits_Z d\mu(z)\ e^{-\nu(z|z)} |\psi(z)|^2 < +\infty.$$

Here $(z|w)$ denotes the K-invariant scalar product on Z normalized by $(e_1|e_1) = 1$ for all minimal tripotents $e_1 \in Z$, and the "invariant" measure is

$$d\mu(z) = \nu^d\ \frac{dz}{\pi^d}.$$

Here d is the (complex) dimension of Z and dz is the Lebesgue measure for the inner product. The reproducing kernel of $H^2_\nu(Z)$ is

$$\mathcal{K}(z,w) = e^{\nu(z|w)}.$$

Accordingly, we have

$$(\mathcal{B}f)(z) = \int\limits_Z d\mu(w)\ \frac{e^{\nu(z|w)}\ e^{\nu(w|z)}}{e^{\nu(z|z)}\ e^{\nu(w|w)}}\ f(w) = \nu^d \int\limits_Z \frac{dw}{\pi^d}\ e^{-\nu(z-w|z-w)} f(w).$$

In particular,

$$(\mathcal{B}f)(0) = \nu^d \int\limits_Z \frac{dw}{\pi^d}\ e^{-\nu(w|w)} f(w).$$

The basic numerical invariants of an irreducible hermitian Jordan triple Z of rank r can be described via the *Peirce decomposition*

$$Z = U \times V = X^{\mathbb{C}} \times V$$

with respect to a maximal tripotent $e \in Z$ of rank r. Here the Peirce 1-space U is the complexification of a Euclidean Jordan algebra X with unit element e, and the

Peirce $\frac{1}{2}$-space V carries a Jordan algebra representation of X. The "characteristic multiplicities" a and b are defined by

$$\begin{aligned} d_X := \dim_{\mathbb{R}}\, X = \dim_{\mathbb{C}}\, U = r + \frac{a}{2}\, r(r-1) \\ \dim_{\mathbb{C}}\, V = rb. \end{aligned} \tag{1}$$

Hence

$$\dim_{\mathbb{C}}\, Z = d = r + \frac{a}{2}\; r(r-1) + rb.$$

The *genus* p of Z is defined by

$$p = \frac{2\,\dim U + \dim V}{r} = 2 + a(r-1) + b.$$

The *Jordan triple determinant* $h(z,w)$ is a (non-homogeneous) sesqui-polynomial on $Z \times \overline{Z}$ whose pth power coincides with the determinant of the so-called Bergman operator $B(z,w)$ acting on Z [L]. For the matrix space $Z = \mathbb{C}^{r\times s}$ of genus $p = r+s$, with triple product $\{uv^*w\} = (uv^*w + wv^*u)/2$, the Bergman operator is given by $B(z,w)\,v = (\mathbf{1} - zw^*)\,v(\mathbf{1} - w^*z)$ and $h(z,w) = \det(\mathbf{1} - zw^*)$, where $\mathbf{1}$ denotes the unit matrix.

For $Z^\bullet = Z^-$, the *non-compact* case, ν is a real parameter $> p-1$ and $H^2_\nu(Z^-)$ is the (weighted) Bergman space of holomorphic functions $\psi \in \mathcal{O}(Z^-)$ satisfying

$$\|\psi\|^2_\nu = \int\limits_{Z^-} d\mu^-(z)\, h(z,z)^\nu |\psi(z)|^2 < +\infty,$$

for the G^--invariant measure

$$d\mu^-(z) = \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega(\nu - \frac{d}{r})}\; \frac{dz}{\pi^d}\; h(z,z)^{-p}.$$

Here

$$\Gamma_\Omega(\lambda) = (2\pi)^{\frac{d_X - r}{2}} \prod_{j=1}^{r} \Gamma(\lambda_j - \frac{a}{2}\,(j-1))$$

is the Gindikin Γ-function of the symmetric cone Ω of X [FK], [G]. The reproducing kernel of $H^2_\nu(Z^-)$ is given by

$$\mathcal{K}^-(z,w) = h(z,w)^{-\nu}.$$

Therefore

$$\begin{aligned} (\mathcal{B}^- f)(z) &= \int\limits_{Z^-} d\mu^-(w)\; \frac{h(z,z)^\nu h(w,w)^\nu}{h(z,w)^\nu h(w,z)^\nu}\; f(w) \\ &= \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega(\nu - \frac{d}{r})} \int\limits_{Z^-} \frac{dw}{\pi^d}\; \frac{h(z,z)^\nu h(w,w)^{\nu - p}}{h(z,w)^\nu h(w,z)^\nu}\; f(w). \end{aligned}$$

In particular

$$(\mathcal{B}^- f)(0) = \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega(\nu - \frac{d}{r})} \int\limits_{Z^-} \frac{dw}{\pi^d}\ h(w,w)^{\nu-p} f(w).$$

For $Z^\bullet = Z^+$, the *compact* case, $\nu \in \mathbb{N}$ is a non-negative integer and, as observed in [Z2], $H^2_\nu(Z^+)$ can be realized as a Bergman type space of entire functions $\psi \in \mathcal{O}(Z)$ satisfying

$$\|\psi\|^2_\nu = \int\limits_Z d\mu^+(z)\, h(z,-z)^{-\nu}\, |\psi(z)|^2 < +\infty$$

for the G^+-invariant measure

$$d\mu^+(z) = \frac{\Gamma_\Omega(\nu+p)}{\Gamma_\Omega(\nu+p-\frac{d}{r})}\ \frac{dz}{\pi^d}\ h(z,-z)^{-p}.$$

This space is finite-dimensional, as can be seen from the reproducing kernel

$$\mathcal{K}^+(z,w) = h(z,-w)^\nu.$$

Therefore

$$\begin{aligned}(\mathcal{B}^+ f)(z) &= \int\limits_Z d\mu^+(w)\ \frac{h(z,-w)^\nu\, h(w,-z)^\nu}{h(z,-z)^\nu\, h(w,-w)^\nu}\ f(w)\\ &= \frac{\Gamma_\Omega(\nu+p)}{\Gamma_\Omega(\nu+p-\frac{d}{r})} \int\limits_Z \frac{dw}{\pi^d}\ \frac{h(z,-w)^\nu\, h(w,-z)^\nu}{h(z,-z)^\nu\, h(w,-w)^{\nu+p}}\ f(w).\end{aligned}$$

In particular,

$$(\mathcal{B}^+ f)(0) = \frac{\Gamma_\Omega(\nu+p)}{\Gamma_\Omega(\nu+p-\frac{d}{r})} \int\limits_Z \frac{dw}{\pi^d}\ h(w,-w)^{-(\nu+p)} f(w).$$

This concludes our case-by-case discussion.

By [U], [FK], the polynomial algebra $\mathcal{P}(Z)$ over Z has a *Peter-Weyl decomposition*

$$\mathcal{P}(Z) = \sum_{\boldsymbol{m}\in\mathbb{N}^r_+} \mathcal{P}_{\boldsymbol{m}}(Z)$$

under the natural K-action. Here $\mathbb{N}^r_+$ denotes the set of all integer partitions

$$\boldsymbol{m} = (m_1, \ldots, m_r)$$

of length $\leq r$. We have a corresponding expansion

$$e^{(z|w)} = \sum_{\boldsymbol{m}} E^{\boldsymbol{m}}(z,w)$$

of the Fischer-Fock kernel in terms of the reproducing kernels $E^{\boldsymbol{m}}(z,w)$ of the finite-dimensional subspaces $\mathcal{P}_{\boldsymbol{m}}(Z)$. These functions are polynomials on $Z \times \overline{Z}$ and therefore give rise to constant coefficient bi-differential operators $E^{\boldsymbol{m}}(\partial,\partial)$, where ∂ denotes the complex Wirtinger derivative. Specifically, for any fixed $u,v\in Z$,

$$E^{\boldsymbol{m}}(\partial,\partial)\,e^{(z|v)+(u|w)}=E^{\boldsymbol{m}}(u,v)\,e^{(z|v)+(u|w)}.$$

Via the diagonal embedding $z\mapsto(z,\overline{z})$ of Z into $Z\times\overline{Z}$, we also have the operators $\partial_{E^{\boldsymbol{m}}}$ acting on Z by

$$\partial_{E^{\boldsymbol{m}}}e^{(z|v)+(u|z)}=E^{\boldsymbol{m}}(u,v)\,e^{(z|v)+(u|z)}.$$

Theorem 2.1. *In terms of the bi-differential operators $\partial_{E^{\boldsymbol{m}}}$ on Z, the Berezin transform has the asymptotic expansion*

$$(\mathcal{B}^{\bullet}f)(0)=\sum_{\boldsymbol{m}}c^{\bullet}_{\boldsymbol{m}}(\nu)(\partial_{E^{\boldsymbol{m}}}f)(0)$$

at 0, *with $\boldsymbol{m}=(m_1,\ldots,m_r)$ running over all partitions of length $\leq r$. Here the coefficients are given by*

$$c_{\boldsymbol{m}}(\nu)=\frac{1}{\nu^{|\boldsymbol{m}|}}\qquad\text{(flat case)}$$

$$c^{-}_{\boldsymbol{m}}(\nu)=\frac{1}{(\nu)_{\boldsymbol{m}}}\qquad\text{(non-compact case)}$$

$$c^{+}_{\boldsymbol{m}}(\nu)=\left(\nu+p-\frac{d}{r}\right)_{-\boldsymbol{m}^*}\qquad\text{(compact case)}$$

where $\boldsymbol{m}^*:=(m_r,\ldots,m_1)$.

The asymptotic expansion above holds in the usual sense that

$$(\mathcal{B}^{\bullet}f)(0)-\sum_{|\boldsymbol{m}|<n}c^{\bullet}_{\boldsymbol{m}}(\nu)(\partial_{E^{\boldsymbol{m}}}f)(0)=O(\nu^{-n})\qquad\text{as }\nu\to+\infty,$$

for all $n=0,1,2,\ldots$. However, and more importantly, this expansion represents also the Peter-Weyl decomposition of the K-invariant linear functional $f\mapsto(\mathcal{B}^{\bullet}f)(0)$ into its components under the natural action of the isotropy subgroup K. The asymptotic expansion in Theorem 2.1 in terms of K-invariant functionals leads in the standard way to the asymptotic expansion of the Berezin transform in terms of $G^{\bullet}$-invariant operators. Similar comments apply to Theorem 3.3 below.

The reader is referred to the paper [EU2] for more details, as well as for applications to quantization (star products) on symmetric spaces.

Proof. For K-invariant integrable functions f on Z we have the polar integration formula [AU2, Proposition 3.4]

$$\int_Z\frac{dx}{\pi^d}\,f(z)=\int_\Omega\frac{dx}{\Gamma_\Omega\left(\frac{d}{r}\right)}\,\Delta(x)^{\frac{d}{r}-\frac{d_X}{r}}f(\sqrt{x}),$$

where Δ is the *Jordan algebra determinant* of X [FK]. For $x\in\Omega$, we have

$$\frac{E^{\boldsymbol{m}}(\sqrt{x},\sqrt{x})}{E^{\boldsymbol{m}}(e,e)}=\phi^{\boldsymbol{m}}(x)$$

where $\phi^{\boldsymbol{m}}$ is the spherical polynomial of type $\boldsymbol{m}$. Let Δ^α be the conical function on Ω, for $\alpha = (\alpha_1, \ldots, \alpha_r)$. By [FK, Theorem VII.1.7] we have

$$\int\limits_{\Omega\cap(e-\Omega)} dx\, \Delta^\alpha(x)\ \Delta^\beta(e-x) = \frac{\Gamma_\Omega(\alpha + \frac{d_X}{r})\ \Gamma_\Omega(\beta + \frac{d_X}{r})}{\Gamma_\Omega(\alpha+\beta+2\frac{d_X}{r})}, \tag{2}$$

and [G, Proposition 2.6] or [AU1, Lemma 5.7] imply

$$\int\limits_\Omega dx\, \Delta^\alpha(x)\, \Delta^{-\beta}(e+x) = \frac{\Gamma_\Omega(\alpha + \frac{d_X}{r})\ \Gamma_\Omega(\beta - \alpha^* - \frac{d_X}{r})}{\Gamma_\Omega(\beta)}$$

with $\alpha^* = (\alpha_r, \ldots, \alpha_1)$. In the *flat* case we have

$$e^{-\nu(\sqrt{x}|\sqrt{x})} = e^{-\nu(x|e)} = e^{-(x|\nu e)}$$

for $x \in \Omega$. This yields

$$\begin{aligned}
\int\limits_Z \frac{dw}{\pi^d}\ e^{-\nu(w|w)}\ \frac{E^{\boldsymbol{m}}(w,w)}{E^{\boldsymbol{m}}(e,e)} &= \int\limits_\Omega \frac{dx}{\Gamma_\Omega(\frac{d}{r})}\ \Delta(x)^{\frac{d}{r}-\frac{d_X}{r}}\ e^{-\nu(\sqrt{x}|\sqrt{x})}\ \frac{E^{\boldsymbol{m}}(\sqrt{x},\sqrt{x})}{E^{\boldsymbol{m}}(e,e)} \\
&= \int\limits_\Omega \frac{dx}{\Gamma_\Omega(\frac{d}{r})}\ \Delta(x)^{\frac{d}{r}-\frac{d_X}{r}}\ e^{-\nu(x|e)}\ \phi^{\boldsymbol{m}}(x) \\
&= \int\limits_\Omega \frac{dx}{\Gamma_\Omega(\frac{d}{r})}\ \Delta(x)^{\frac{d}{r}-\frac{d_X}{r}}\ e^{-\nu(x|e)}\ \Delta^{\boldsymbol{m}}(x) \\
&= \int\limits_\Omega \frac{dx}{\Gamma_\Omega(\frac{d}{r})}\ \Delta^{\boldsymbol{m}+\frac{d}{r}-\frac{d_X}{r}}(x)\ e^{-(x|\nu e)} \\
&= \frac{\Gamma_\Omega(\boldsymbol{m}+\frac{d}{r})}{\Gamma_\Omega(\frac{d}{r})}\ \Delta^{\boldsymbol{m}+\frac{d}{r}}\ (\nu^{-1}e) \\
&= \left(\frac{d}{r}\right)_{\boldsymbol{m}}\ \nu^{-|m|-d},
\end{aligned}$$

since $\Delta^{\boldsymbol{m}+\frac{d}{r}}$ is homogeneous of total degree $|\boldsymbol{m}|+d$. (The third equality in the chain uses the fact that dx, $\Delta(x)$ and $e^{(x|e)}$ are all invariant under the subgroup $L \subset \mathrm{Aut}(\Omega)$ stabilizing e, while $\phi^{\boldsymbol{m}}$ is the average of $\Delta^{\boldsymbol{m}}$ over L.) Therefore

$$\frac{(\mathcal{B}E^{\boldsymbol{m}})(0)}{E^{\boldsymbol{m}}(e,e)} = \nu^d \int\limits_Z \frac{dw}{\pi^d}\ e^{-\nu(w|w)}\ \frac{E^{\boldsymbol{m}}(w,w)}{E^{\boldsymbol{m}}(e,e)} = \nu^{-|m|}\ \left(\frac{d}{r}\right)_{\boldsymbol{m}}.$$

In the *non-compact* case we have

$$h(\sqrt{x},\sqrt{x}) = \Delta(e-x)$$

for $x \in \Omega \cap (e-\Omega)$. This yields

$$\begin{aligned}
&\int\limits_{Z^-} \frac{dw}{\pi^d}\ h(w,w)^{\nu-p}\ \frac{E^{\boldsymbol{m}}(w,w)}{E^{\boldsymbol{m}}(e,e)} \\
&\qquad = \int\limits_{\Omega\cap(e-\Omega)} \frac{dx}{\Gamma_\Omega(\frac{d}{r})}\ \Delta(x)^{\frac{d}{r}-\frac{d_X}{r}}\ h(\sqrt{x},\sqrt{x})^{\nu-p}\ \frac{E^{\boldsymbol{m}}(\sqrt{x},\sqrt{x})}{E^{\boldsymbol{m}}(e,e)} \\
&\qquad = \int\limits_{\Omega\cap(e-\Omega)} \frac{dx}{\Gamma_\Omega(\frac{d}{r})}\ \Delta(x)^{\frac{d}{r}-\frac{d_X}{r}}\ \Delta(e-x)^{\nu-p}\ \phi^{\boldsymbol{m}}(x) \\
&\qquad = \int\limits_{\Omega\cap(e-\Omega)} \frac{dx}{\Gamma_\Omega(\frac{d}{r})}\ \Delta(x)^{\frac{d}{r}-\frac{d_X}{r}}\ \Delta(e-x)^{\nu-p}\,\Delta^{\boldsymbol{m}}(x) \\
&\qquad = \int\limits_{\Omega\cap(e-\Omega)} \frac{dx}{\Gamma_\Omega(\frac{d}{r})}\ \Delta^{\boldsymbol{m}+\frac{d}{r}-\frac{d_X}{r}}(x)\ \Delta(e-x)^{\nu-p} = \frac{\Gamma_\Omega(\boldsymbol{m}+\frac{d}{r})}{\Gamma_\Omega(\frac{d}{r})}\ \frac{\Gamma_\Omega(\nu-\frac{d}{r})}{\Gamma_\Omega(\nu+\boldsymbol{m})}
\end{aligned}$$

since $\nu - p + \frac{d_X}{r} = \nu - \frac{d}{r}$. Therefore

$$\begin{aligned}
\frac{(\mathcal{B}^- E^{\boldsymbol{m}})(0)}{E^{\boldsymbol{m}}(e,e)} &= \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega(\nu-\frac{d}{r})} \int\limits_{Z^-} \frac{dw}{\pi^d}\ h(w,w)^{\nu-p}\ \frac{E^{\boldsymbol{m}}(w,w)}{E^{\boldsymbol{m}}(e,e)} \\
&= \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega(\nu-\frac{d}{r})}\ \frac{\Gamma_\Omega(\boldsymbol{m}+\frac{d}{r})}{\Gamma_\Omega(\frac{d}{r})}\ \frac{\Gamma_\Omega(\nu-\frac{d}{r})}{\Gamma_\Omega(\nu+\boldsymbol{m})} = \frac{(d/r)_{\boldsymbol{m}}}{(\nu)_{\boldsymbol{m}}}.
\end{aligned}$$

In the *compact* case, we have

$$h(\sqrt{x},-\sqrt{x}) = \Delta(e+x)$$

for $x \in \Omega$. This yields

$$\begin{aligned}
&\int\limits_{Z} \frac{dw}{\pi^d}\ h(w,-w)^{-(\nu+p)}\ \frac{E^{\boldsymbol{m}}(w,w)}{E^{\boldsymbol{m}}(e,e)} \\
&\qquad = \int\limits_{\Omega} \frac{dx}{\Gamma_\Omega(\frac{d}{r})}\ \Delta(x)^{\frac{d}{r}-\frac{d_X}{r}}\ h(\sqrt{x},-\sqrt{x})^{-(\nu+p)}\ \frac{E^{\boldsymbol{m}}(\sqrt{x},\sqrt{x})}{E^{\boldsymbol{m}}(e,e)} \\
&\qquad = \int\limits_{\Omega} \frac{dx}{\Gamma_\Omega(\frac{d}{r})}\ \Delta(x)^{\frac{d}{r}-\frac{d_X}{r}}\ \Delta(e+x)^{-(\nu+p)}\ \phi^{\boldsymbol{m}}(x) \\
&\qquad = \int\limits_{\Omega} \frac{dx}{\Gamma_\Omega(\frac{d}{r})}\ \Delta(x)^{\frac{d}{r}-\frac{d_X}{r}}\,\Delta(e+x)^{-(\nu+p)}\,\Delta^{\boldsymbol{m}}(x) \\
&\qquad = \int\limits_{\Omega} \frac{dx}{\Gamma_\Omega(\frac{d}{r})}\ \Delta^{\boldsymbol{m}+\frac{d}{r}-\frac{d_X}{r}}(x)\ \Delta(e+x)^{-(\nu+p)}
\end{aligned}$$

$$= \frac{\Gamma_\Omega(\boldsymbol{m} + \frac{d}{r})\,\Gamma_\Omega(\nu + p - \boldsymbol{m}^* - \frac{d}{r})}{\Gamma_\Omega(\frac{d}{r})\,\Gamma_\Omega(\nu + p)}.$$

Therefore

$$\frac{(\mathcal{B}^+ E^{\boldsymbol{m}})(0)}{E^{\boldsymbol{m}}(e,e)} = \frac{\Gamma_\Omega(\nu+p)}{\Gamma_\Omega(\nu+p-\frac{d}{r})} \int_Z \frac{dw}{\pi^d}\, h(w,-w)^{-(\nu+p)}\, \frac{E^{\boldsymbol{m}}(w,w)}{E^{\boldsymbol{m}}(e,e)}$$

$$= \frac{\Gamma_\Omega(\nu+p)}{\Gamma_\Omega(\nu+p-\frac{d}{r})}\; \frac{\Gamma_\Omega(\boldsymbol{m} + \frac{d}{r})\,\Gamma_\Omega(\nu + p - \boldsymbol{m}^* - \frac{d}{r})}{\Gamma_\Omega(\frac{d}{r})\,\Gamma_\Omega(\nu + p)}$$

$$= (d/r)_{\boldsymbol{m}} \left(\nu + p - \frac{d}{r}\right)_{-\boldsymbol{m}^*}.$$

In all three cases, since $\mathcal{B}^\bullet$ is a $G^\bullet$-invariant operator on $Z^\bullet$, the localized operator at 0 has a unique expansion

$$(\mathcal{B}^\bullet f)(0) = \sum_{\boldsymbol{m}} c^\bullet_{\boldsymbol{m}}(\nu)\,(\partial_{E^{\boldsymbol{m}}} f)(0)$$

for all functions f which are smooth near $0 \in Z$. This implies for the diagonal $E^{\boldsymbol{m}}(z,z)$

$$(\mathcal{B}^\bullet E^{\boldsymbol{m}})(0) = \sum_{\boldsymbol{n}} c^\bullet_{\boldsymbol{n}}(\nu)\,\partial_{E^{\boldsymbol{n}}} E^{\boldsymbol{m}}(0)$$

$$= c^\bullet_{\boldsymbol{m}}(\nu)\, \|E^{\boldsymbol{m}}\|^2_{Z\times\overline{Z}} = c^\bullet_{\boldsymbol{m}}(\nu)\; E^{\boldsymbol{m}}(e,e)\, \left(\frac{d}{r}\right)_{\boldsymbol{m}}.$$

It follows that

$$c^\bullet_{\boldsymbol{m}}(\nu) = \frac{1}{(d/r)_{\boldsymbol{m}}}\; \frac{\mathcal{B}^\bullet E^{\boldsymbol{m}}(0)}{E^{\boldsymbol{m}}(e,e)}$$

has the value specified above. □

Remark 2.2. The non-compact case of the last theorem recovers the result of [AO].

3. Berezin transform for real symmetric spaces

In order to introduce the "real" counterparts of hermitian symmetric spaces, let $Z_{\mathbb{C}}$ be a hermitian Jordan triple endowed with a triple involution $z \mapsto \tilde{z}$. Then the real form

$$Z_{\mathbb{R}} = \{z \in Z_{\mathbb{C}} : \tilde{z} = z\}$$

is a Euclidean Jordan triple which we assume to be *irreducible*. The associated unit ball $Z^-_{\mathbb{R}} \subset Z_{\mathbb{R}}$ and compact dual $Z^+_{\mathbb{R}} \supset Z_{\mathbb{R}}$ have the hermitifications $Z^-_{\mathbb{C}}$ and $Z^+_{\mathbb{C}}$, respectively. In summary,

$$\begin{array}{ccccc} Z^-_{\mathbb{C}} & \subset & Z_{\mathbb{C}} & \subset & Z^+_{\mathbb{C}} \\ \cup & & \cup & & \cup \\ Z^-_{\mathbb{R}} & \subset & Z_{\mathbb{R}} & \subset & Z^+_{\mathbb{R}}. \end{array} \tag{3}$$

Remark 3.1. As a special case of this situation, we obtain the "product" case

$$Z_{\mathbb{R}} := Z_{\mathrm{diag}} = \{(z,\overline{z}) : z \in Z\} \quad \subset Z_{\mathbb{C}} := Z \times \overline{Z},$$
$$Z_{\mathbb{R}}^{-} := Z_{\mathrm{diag}}^{-} = \{(z,\overline{z}) : z \in Z^{-}\} \subset Z_{\mathbb{C}}^{-} := Z^{-} \times \overline{Z}^{-},$$
$$Z_{\mathbb{R}}^{+} := Z_{\mathrm{diag}}^{+} = \{(z,\overline{z}) : z \in Z^{+}\} \subset Z_{\mathbb{C}}^{+} := Z^{+} \times \overline{Z}^{+},$$

associated with a hermitian Jordan triple Z and the "flip" involution $(z,\overline{w})^{\sim} := (w,\overline{z})$. In this case (3) takes the form

$$\begin{array}{ccccc} Z^{-} \times \bar{Z}^{-} & \subset & Z \times \bar{Z} & \subset & Z^{+} \times \bar{Z}^{+} \\ \cup & & \cup & & \cup \\ Z_{\mathrm{diag}}^{-} & \subset & Z_{\mathrm{diag}} & \subset & Z_{\mathrm{diag}}^{+}. \end{array}$$

This is the only case where the complexified spaces are not irreducible [L]. Since this situation is covered by Section 2, we will assume from now on that $Z_{\mathbb{C}}$ is irreducible.

There exists a maximal tripotent $e \in Z_{\mathbb{R}}$ (of rank $r_{\mathbb{R}}$) which is also maximal in $Z_{\mathbb{C}}$ (i.e., of rank $r_{\mathbb{C}}$). Let

$$Z_{\mathbb{C}} = U_{\mathbb{C}} \times V_{\mathbb{C}} = X_{\mathbb{C}}^{\mathbb{C}} \times V_{\mathbb{C}}$$

and

$$Z_{\mathbb{R}} = U_{\mathbb{R}} \times V_{\mathbb{R}} = X_{\mathbb{R}} \times Y_{\mathbb{R}} \times V_{\mathbb{R}}$$

denote the corresponding Peirce decompositions. Then we have complexifications

$$U_{\mathbb{C}} = U_{\mathbb{R}}^{\mathbb{C}}, \quad V_{\mathbb{C}} = V_{\mathbb{R}}^{\mathbb{C}}$$

and the Euclidean Jordan algebras $X_{\mathbb{C}}$ (of rank $r_{\mathbb{C}}$) and $X_{\mathbb{R}}$ (of rank $r_{\mathbb{R}}$) are related by

$$X_{\mathbb{R}} = \{x \in X_{\mathbb{C}} : \tilde{x} = x\},$$
$$iY_{\mathbb{R}} = \{y \in X_{\mathbb{C}} : \tilde{y} = -y\}.$$

Equivalently,

$$X_{\mathbb{R}} = \{x \in U_{\mathbb{R}} : x^{*} = x\},$$
$$Y_{\mathbb{R}} = \{y \in U_{\mathbb{R}} : y^{*} = -y\}$$

for the Jordan involution * in $U_{\mathbb{C}}$. Writing e as an orthogonal sum of minimal tripotents in $Z_{\mathbb{R}}$ and $Z_{\mathbb{C}}$, respectively, it follows that the normalized inner products $(x|y)_{\mathbb{R}}$ in $Z_{\mathbb{R}}$ and $(z|w)_{\mathbb{C}}$ in $Z_{\mathbb{C}}$ satisfy $(e|e)_{\mathbb{R}} = r_{\mathbb{R}}$ and $(e|e)_{\mathbb{C}} = r_{\mathbb{C}}$. Therefore we have the reciprocity

$$r_{\mathbb{R}}(x|y)_{\mathbb{C}} = r_{\mathbb{C}}(x|y)_{\mathbb{R}}$$

for all $x, y \in Z_{\mathbb{R}} \subset Z_{\mathbb{C}}$. Analogous to (1), we have characteristic multiplicities $a_{\mathbb{C}}, b_{\mathbb{C}}$ satisfying

$$\dim_{\mathbb{R}} V_{\mathbb{R}} = \dim_{\mathbb{C}} V_{\mathbb{C}} = r_{\mathbb{C}}\, b_{\mathbb{C}}$$
$$\dim_{\mathbb{R}} U_{\mathbb{R}} = \dim_{\mathbb{C}} U_{\mathbb{C}} = r_{\mathbb{C}} + \frac{a_{\mathbb{C}}}{2}\, r_{\mathbb{C}}\,(r_{\mathbb{C}} - 1)$$

and hence

$$\dim_{\mathbb{R}} Z_{\mathbb{R}} = \dim_{\mathbb{C}} Z_{\mathbb{C}} = r_{\mathbb{C}} + \frac{a_{\mathbb{C}}}{2} r_{\mathbb{C}} (r_{\mathbb{C}} - 1) + r_{\mathbb{C}} b_{\mathbb{C}}.$$

Also, the "complex" genus is

$$p_{\mathbb{C}} = \frac{2 \dim_{\mathbb{C}} U_{\mathbb{C}} + \dim_{\mathbb{C}} V_{\mathbb{C}}}{r_{\mathbb{C}}} = 2 + a_{\mathbb{C}}(r_{\mathbb{C}} - 1) + b_{\mathbb{C}}.$$

In the real case, we define $a_{\mathbb{R}}, b_{\mathbb{R}}$ via

$$\dim_{\mathbb{R}} X_{\mathbb{R}} = r_{\mathbb{R}} + \frac{a_{\mathbb{R}}}{2} r_{\mathbb{R}} (r_{\mathbb{R}} - 1)$$
$$\dim_{\mathbb{R}} V_{\mathbb{R}} = r_{\mathbb{R}} b_{\mathbb{R}}$$

and introduce another numerical invariant $c_{\mathbb{R}}$ via

$$\dim_{\mathbb{R}} Y_{\mathbb{R}} = r_{\mathbb{R}} c_{\mathbb{R}} + \frac{a_{\mathbb{R}}}{2} r_{\mathbb{R}} (r_{\mathbb{R}} - 1).$$

One can show that this covers all cases of the classification [L], [Z1], [EU1] with one exception (type D_2) which will not be considered here. The "real" genus $p_{\mathbb{R}}$ is defined by

$$p_{\mathbb{R}} = \frac{p_{\mathbb{C}} r_{\mathbb{C}}}{2r_{\mathbb{R}}} = \frac{\dim_{\mathbb{R}} U_{\mathbb{R}}}{r_{\mathbb{R}}} + \frac{\dim_{\mathbb{R}} V_{\mathbb{R}}}{2r_{\mathbb{R}}} = 1 + c_{\mathbb{R}} + a_{\mathbb{R}}(r_{\mathbb{R}} - 1) + \frac{b_{\mathbb{R}}}{2}.$$

In terms of $d_Y = \dim_{\mathbb{R}} Y_{\mathbb{R}}$ and $d_X = \dim_{\mathbb{R}} X_{\mathbb{R}} = \dim_{\mathbb{R}} X_{\mathbb{C}} - d_Y$, we have the relations

$$p_{\mathbb{R}} - \frac{d}{2r_{\mathbb{R}}} = \frac{d_X + d_Y}{2r_{\mathbb{R}}}$$

and hence

$$\frac{d}{2r_{\mathbb{R}}} + \frac{d_X}{r_{\mathbb{R}}} - p_{\mathbb{R}} = \frac{d_X - d_Y}{2r_{\mathbb{R}}}.$$

Since $Z_{\mathbb{C}}$ is irreducible, we may consider the quantization Hilbert spaces $H^2_{\nu_{\mathbb{C}}}(Z^\bullet_{\mathbb{C}})$ introduced in Section 1, for the appropriate range of parameters $\nu_{\mathbb{C}}$, with reproducing kernel denoted by $\mathcal{K}^\bullet_{\mathbb{C}}(z,w)$ for $z, w \in Z^\bullet_{\mathbb{C}}$. Besides the "complex" Berezin transform $\mathcal{B}^\bullet_{\mathbb{C}}$ on $L^2(Z^\bullet_{\mathbb{C}}, d\mu^\bullet_{\mathbb{C}})$ defined as above, for the "invariant" measure $d\mu^\bullet_{\mathbb{C}}$ normalized as in Section 2, there also exists a "real" Berezin transform $\mathcal{B}^\bullet_{\mathbb{R}}$, which is a densely defined self-adjoint operator on $L^2(Z^\bullet_{\mathbb{R}}, d\mu^\bullet_{\mathbb{R}})$, with integral representation

$$(\mathcal{B}^\bullet_{\mathbb{R}} f)(z) = \int\limits_{Z^\bullet_R} d\mu^\bullet_{\mathbb{R}}(w) \; \frac{\mathcal{K}^\bullet_{\mathbb{C}}(z,w)}{\mathcal{K}^\bullet_{\mathbb{C}}(z,z)^{1/2} \, \mathcal{K}^\bullet_{\mathbb{C}}(w,w)^{1/2}} \; f(w).$$

Here we use the fact that $\mathcal{K}^\bullet_{\mathbb{C}}(z,z) > 0$ for all $z \in Z^\bullet_{\mathbb{R}}$. For motivation and background concerning the real Berezin transform, cf. [AU1], [Z1]. We define the parameter $\nu_{\mathbb{R}}$ by the condition

$$\nu_{\mathbb{C}} \, r_{\mathbb{C}} = 2\nu_{\mathbb{R}} \, r_{\mathbb{R}}.$$

As in Section 1, we will now discuss the three types of real symmetric spaces separately. For $Z^\bullet_{\mathbb{R}} = Z_{\mathbb{R}}$, the real *flat case*, the invariant measure is

$$d\mu_{\mathbb{R}}(z) = \nu_{\mathbb{R}}^{d/2} \, \frac{dz}{\pi^{d/2}}.$$

Therefore

$$(\mathcal{B}_{\mathbb{R}}f)(z) = \int_{Z_{\mathbb{R}}} d\mu_{\mathbb{R}}(w)\ \frac{e^{\nu_{\mathbb{C}}(z|w)_{\mathbb{C}}}}{e^{\frac{\nu_{\mathbb{C}}}{2}(z|z)_{\mathbb{C}}}\, e^{\frac{\nu_{\mathbb{C}}}{2}(w|w)_{\mathbb{C}}}}\ f(w)$$

$$= \nu_{\mathbb{R}}^{d/2} \int_{Z_{\mathbb{R}}} \frac{dw}{\pi^{d/2}}\ \frac{e^{\nu_{\mathbb{C}}(z|w)_{\mathbb{C}}}}{e^{\frac{\nu_{\mathbb{C}}}{2}(z|z)_{\mathbb{C}}}\, e^{\frac{\nu_{\mathbb{C}}}{2}(w|w)_{\mathbb{C}}}}\ f(w).$$

In particular,

$$(\mathcal{B}_{\mathbb{R}}f)(0) = \nu_{\mathbb{R}}^{d/2} \int_{Z_{\mathbb{R}}} \frac{dw}{\pi^{d/2}}\ e^{-\frac{\nu_{\mathbb{C}}}{2}(w|w)_{\mathbb{C}}}\, f(w).$$

For $Z_{\mathbb{R}}^{\bullet} = Z_{\mathbb{R}}^{-}$, the real *non-compact* case, the invariant measure is

$$d\mu_{\mathbb{R}}^{-}(z) = \frac{\Gamma_{\Omega}(\nu_{\mathbb{R}} + \frac{d_X - d_Y}{2r})}{\Gamma_{\Omega}(\nu_{\mathbb{R}} - p_{\mathbb{R}} + \frac{d_X}{r})}\ \frac{dz}{\pi^{d/2}}\ h_{\mathbb{C}}(z,z)^{-p_{\mathbb{C}}/2},$$

where $h_{\mathbb{C}}$ denotes the Jordan triple determinant of $Z_{\mathbb{C}}$. Therefore we obtain

$$(\mathcal{B}_{\mathbb{R}}^{-}f)(z) = \int_{Z_{\mathbb{R}}^{-}} d\mu_{\mathbb{R}}^{-}(w)\ \frac{h_{\mathbb{C}}(z,z)^{\nu_{\mathbb{C}}/2}\, h_{\mathbb{C}}(w,w)^{\nu_{\mathbb{C}}/2}}{h_{\mathbb{C}}(z,w)^{\nu_{\mathbb{C}}}}\ f(w)$$

$$= \frac{\Gamma_{\Omega}(\nu_{\mathbb{R}} + \frac{d_X - d_Y}{2r})}{\Gamma_{\Omega}(\nu_{\mathbb{R}} - p_{\mathbb{R}} + \frac{d_X}{r})} \int_{Z_{\mathbb{R}}^{-}} \frac{dw}{\pi^{d/2}}\ \frac{h_{\mathbb{C}}(z,z)^{\nu_{\mathbb{C}}/2}\, h_{\mathbb{C}}(w,w)^{(\nu_{\mathbb{C}}-p_{\mathbb{C}})/2}}{h_{\mathbb{C}}(z,w)^{\nu_{\mathbb{C}}}}\ f(w).$$

In particular

$$(\mathcal{B}_{\mathbb{R}}^{-}f)(0) = \frac{\Gamma_{\Omega}(\nu_{\mathbb{R}} + \frac{d_X - d_Y}{2r})}{\Gamma_{\Omega}(\nu_{\mathbb{R}} - p_{\mathbb{R}} + \frac{d_X}{r})} \int_{Z_{\mathbb{R}}^{-}} \frac{dw}{\pi^{d/2}}\ h_{\mathbb{C}}(w,w)^{(\nu_{\mathbb{C}}-p_{\mathbb{C}})/2}\, f(w).$$

For $Z_{\mathbb{R}}^{\bullet} = Z_{\mathbb{R}}^{+}$, the real *compact* case, the invariant measure on $Z_{\mathbb{R}}$ is

$$d\mu_{\mathbb{R}}^{+}(z) = \frac{\Gamma_{\Omega}(\nu_{\mathbb{R}} + p_{\mathbb{R}})}{\Gamma_{\Omega}(\nu_{\mathbb{R}} + p_{\mathbb{R}} - \frac{d}{2r})}\ \frac{dz}{\pi^{d/2}}\ h_{\mathbb{C}}(z,-z)^{-p_{\mathbb{C}}/2}.$$

Therefore

$$(\mathcal{B}_{\mathbb{R}}^{+}f)(z) = \int_{Z_{\mathbb{R}}} d\mu_{\mathbb{R}}^{+}(w)\ \frac{h_{\mathbb{C}}(z,-w)^{\nu_{\mathbb{C}}}}{h_{\mathbb{C}}(z,-z)^{\nu_{\mathbb{C}}/2}\ h_{\mathbb{C}}(w,-w)^{(\nu_{\mathbb{C}}+p_{\mathbb{C}})/2}}\ f(w)$$

$$= \frac{\Gamma_{\Omega}(\nu_{\mathbb{R}} + p_{\mathbb{R}})}{\Gamma_{\Omega}(\nu_{\mathbb{R}} + p_{\mathbb{R}} - \frac{d}{2r})} \int_{Z_{\mathbb{R}}} \frac{dw}{\pi^{d/2}}\ \frac{h_{\mathbb{C}}(z,-w)^{\nu_{\mathbb{C}}}}{h_{\mathbb{C}}(z,-z)^{\nu_{\mathbb{C}}/2}\ h_{\mathbb{C}}(w,-w)^{(\nu_{\mathbb{C}}+p_{\mathbb{C}})/2}}\ f(w).$$

In particular

$$(\mathcal{B}_{\mathbb{R}}^{+}f)(0) = \frac{\Gamma_{\Omega}(\nu_{\mathbb{R}} + p_{\mathbb{R}})}{\Gamma_{\Omega}(\nu_{\mathbb{R}} + p_{\mathbb{R}} - \frac{d}{2r})} \int_{Z_{\mathbb{R}}} \frac{dw}{\pi^{d/2}}\ h_{\mathbb{C}}(w,-w)^{-(\nu_{\mathbb{C}}+p_{\mathbb{C}})/2}\, f(w).$$

This concludes our case-by-case discussion.

As in Section 2, the ($\mathbb{C}$-valued) polynomial algebra $\mathcal{P}(X_\mathbb{R}) \equiv \mathcal{P}(X_\mathbb{R}^\mathbb{C})$ has a Peter-Weyl decomposition

$$\mathcal{P}(X_\mathbb{R}^\mathbb{C}) = \sum_{\boldsymbol{m}\in\mathbb{N}_+^{r_\mathbb{R}}} \mathcal{P}_{\boldsymbol{m}}(X_\mathbb{R}^\mathbb{C})$$

under the natural action of $\mathrm{Aut}(X_\mathbb{R}^\mathbb{C})$, and we have a corresponding kernel expansion

$$e^{(x|y)_\mathbb{R}} = \sum_{\boldsymbol{m}\in\mathbb{N}_+^{r_\mathbb{R}}} E_\mathbb{R}^{\boldsymbol{m}}(x,y)$$

for the (irreducible) Euclidean Jordan algebra $X_\mathbb{R}$ of rank $r_\mathbb{R}$, with $\boldsymbol{m}$ running over all partitions of length $r_\mathbb{R}$.

On the other hand, the polynomial algebra $\mathcal{P}(Z_\mathbb{C})$ of the (irreducible) hermitian Jordan triple $Z_\mathbb{C}$ of rank $r_\mathbb{C}$ also admits a Peter-Weyl decomposition

$$\mathcal{P}(Z_\mathbb{C}) = \sum_{\boldsymbol{n}\in\mathbb{N}_+^{r_\mathbb{C}}} \mathcal{P}_{\boldsymbol{n}}(Z_\mathbb{C})$$

under the $K_\mathbb{C}$-action, with $\boldsymbol{n}$ running over all integer partitions of length $r_\mathbb{C}$. Let

$$e^{(z|w)_\mathbb{C}} = \sum_{\boldsymbol{n}\in\mathbb{N}_+^{r_\mathbb{C}}} E_\mathbb{C}^{\boldsymbol{n}}(z,w)$$

denote the corresponding kernel expansion.

A partition $\boldsymbol{n} \in \mathbb{N}_+^{r_\mathbb{C}}$ is called *even* if $\mathcal{P}_{\boldsymbol{n}}(Z_\mathbb{C})$ contains a non-zero $K_\mathbb{R}$-invariant polynomial (which is uniquely determined up to a constant multiple). The results in [Z3] (for tube type domains) and [Z4] (for non-tube type domains) show that with one exception (type A), which we will exclude from consideration here, the even signatures are obtained by "doubling" a signature $\boldsymbol{m} \in \mathbb{N}_+^{r_\mathbb{R}}$ of length $r_\mathbb{R}$. Then the associated $K_\mathbb{R}$-invariant polynomial $E^{\boldsymbol{m}} \in \mathcal{P}_{\boldsymbol{m}_\mathbb{C}}(Z_\mathbb{C})$ is uniquely characterized by the condition

$$E^{\boldsymbol{m}}(x) = E_\mathbb{R}^{\boldsymbol{m}}(x,x) = E_\mathbb{R}^{\boldsymbol{m}}(e,e)\ \phi_\mathbb{R}^{\boldsymbol{m}}(x^2)$$

for all $x \in X_\mathbb{R} \subset X_\mathbb{R}^\mathbb{C} \subset Z_\mathbb{C}$, where $\phi_\mathbb{R}^{\boldsymbol{m}}$ is the spherical polynomial (normalized Jack polynomial) of type $\boldsymbol{m}$ [FK].

Proposition 3.2. *$E^{\boldsymbol{m}}$ has the (complex) Fock space norm*

$$\|E^{\boldsymbol{m}}\|_\mathbb{C}^2 = E_\mathbb{R}^{\boldsymbol{m}}(e,e)\left(\frac{d}{2r_\mathbb{R}}\right)_{\boldsymbol{m}}.$$

Proof. Put $d_{\boldsymbol{m}} := \dim \mathcal{P}_{\boldsymbol{m}}(X_\mathbb{R}^\mathbb{C})$. The Shilov boundary $S \subset X_\mathbb{R}^\mathbb{C}$ is the orbit of e under the group $\mathrm{Aut}(X_\mathbb{R}^\mathbb{C})$. Applying Schur orthogonality and putting $E_v^{\boldsymbol{m}}(u) := E_\mathbb{R}^{\boldsymbol{m}}(u,v)$ for $u,v \in S$, we obtain for the Fock space inner product $(p|q)_\mathbb{R} = (\partial_p\, q)(0)$

on $X_{\mathbb{R}}^{\mathbb{C}}$

$$\frac{E_{\mathbb{R}}^{\boldsymbol{m}}(e,e)^2}{d_{\boldsymbol{m}}} = \frac{(E_e^{\boldsymbol{m}} \mid E_e^{\boldsymbol{m}})_{\mathbb{R}}^2}{d_{\boldsymbol{m}}} = \int_S du\ (E_e^{\boldsymbol{m}} \mid E_u^{\boldsymbol{m}})_{\mathbb{R}}\ (E_u^{\boldsymbol{m}} \mid E_e^{\boldsymbol{m}})_{\mathbb{R}}$$
$$= \int_S du\ |E_e^{\boldsymbol{m}}(u)|^2 = \frac{(E_e^{\boldsymbol{m}} \mid E_e^{\boldsymbol{m}})_{\mathbb{R}}}{(d_X/r_{\mathbb{R}})_{\boldsymbol{m}}} = \frac{E_{\mathbb{R}}^{\boldsymbol{m}}(e,e)}{(d_X/r_{\mathbb{R}})_{\boldsymbol{m}}}.$$

This shows

$$E_{\mathbb{R}}^{\boldsymbol{m}}(e,e) = \frac{d_{\boldsymbol{m}}}{(d_X/r_{\mathbb{R}})_{\boldsymbol{m}}}.$$

By [Z4, Lemma 3.3 and Proposition 3.6] we have for the Fock space inner product on $Z_{\mathbb{C}}$

$$\left\| \frac{E^{\boldsymbol{m}}}{E_{\mathbb{R}}^{\boldsymbol{m}}(e,e)} \right\|_{\mathbb{C}}^2 = \frac{(d_X/r_{\mathbb{R}})_{\boldsymbol{m}}\ (d/2r_{\mathbb{R}})_{\boldsymbol{m}}}{d_{\boldsymbol{m}}}.$$

Therefore

$$\|E^{\boldsymbol{m}}\|_{\mathbb{C}}^2 = E_{\mathbb{R}}^{\boldsymbol{m}}(e,e)^2\ \frac{(d_X/r_{\mathbb{R}})_{\boldsymbol{m}}\ (d/2r_{\mathbb{R}})_{\boldsymbol{m}}}{d_{\boldsymbol{m}}} = E_{\mathbb{R}}^{\boldsymbol{m}}(e,e)\ \left(\frac{d}{2r_{\mathbb{R}}}\right)_{\boldsymbol{m}}. \qquad \square$$

Theorem 3.3. *Consider the (holomorphic) differential operators $\partial_{E^{\boldsymbol{m}}}$ on $Z_{\mathbb{R}} \subset Z_{\mathbb{C}}$ induced by the $K_{\mathbb{R}}$-invariant polynomials $E^{\boldsymbol{m}} \in \mathcal{P}_{\boldsymbol{m}_{\mathbb{C}}}(Z_{\mathbb{C}})$. Then the real Berezin transform has the asymptotic expansion*

$$(\mathcal{B}_{\mathbb{R}}^{\bullet} f)(0) = \sum_{\boldsymbol{m}} c_{\boldsymbol{m}}^{\bullet}(\nu_{\mathbb{R}})\ (\partial_{E^{\boldsymbol{m}}} f)(0),$$

near 0, with $\boldsymbol{m} = (m_1, \ldots, m_{r_{\mathbb{R}}})$ running over all partitions of length $\leq r_{\mathbb{R}}$. Here the coefficients are given by

$$c_{\boldsymbol{m}} = \nu_{\mathbb{R}}^{-|\boldsymbol{m}|} \qquad (\textit{flat case})$$
$$c_{\boldsymbol{m}}^{-} = \frac{1}{\left(\nu_{\mathbb{R}} - p_{\mathbb{R}} + \frac{d}{2r_{\mathbb{R}}} + \frac{d_X}{r_{\mathbb{R}}}\right)_{\boldsymbol{m}}} \qquad (\textit{non-compact case})$$
$$c_{\boldsymbol{m}}^{+} = \left(\nu_{\mathbb{R}} + p_{\mathbb{R}} - \frac{d}{2r_{\mathbb{R}}}\right)_{-\boldsymbol{m}^*} \qquad (\textit{compact case})$$

with $\boldsymbol{m}^* := (m_{r_{\mathbb{R}}}, \ldots, m_1)$.

Proof. By [AU2, Proposition 3.4], one has the polar integration formula

$$\int_{Z_{\mathbb{R}}} \frac{dz}{\pi^{d/2}}\ f(z) = \int_{\Omega} \frac{dx}{\Gamma_{\Omega}\left(\frac{d}{2r_{\mathbb{R}}}\right)}\ \Delta(x)^{\frac{d}{2r_{\mathbb{R}}} - \frac{d_X}{r_{\mathbb{R}}}} f(\sqrt{x})$$

for $K_{\mathbb{R}}$-invariant integrable functions f on $Z_{\mathbb{R}}$, Ω being the cone of $X_{\mathbb{R}}$ (the proof given there, valid for functions supported in $Z_{\mathbb{R}}^{-}$, extends to the general case by a homogeneity and density argument). For $x \in \Omega$, we have

$$E^{\boldsymbol{m}}(\sqrt{x}) = E_{\mathbb{R}}^{\boldsymbol{m}}(\sqrt{x}, \sqrt{x}) = E_{\mathbb{R}}^{\boldsymbol{m}}(e,e)\ \phi_{\mathbb{R}}^{\boldsymbol{m}}(x).$$

In the *flat* case, we have

$$e^{-\frac{\nu_{\mathbb{C}}}{2}(\sqrt{x}|\sqrt{x})_{\mathbb{C}}} = e^{-\nu_{\mathbb{R}}(\sqrt{x}|\sqrt{x})_{\mathbb{R}}} = e^{-(x|\nu_{\mathbb{R}}\,e)_{\mathbb{R}}}$$

for $x \in \Omega$. This yields

$$\begin{aligned}
&\int\limits_{Z_{\mathbb{R}}} \frac{dw}{\pi^{d/2}}\ e^{-\frac{\nu_{\mathbb{C}}}{2}(w|w)_{\mathbb{C}}}\ \frac{E^{\boldsymbol{m}}(w)}{E^{\boldsymbol{m}}_{\mathbb{R}}(e,e)} \\
&\quad = \int\limits_{\Omega} \frac{dx}{\Gamma_\Omega\left(\frac{d}{2r_{\mathbb{R}}}\right)}\ \Delta(x)^{\frac{d}{2r_{\mathbb{R}}}-\frac{d_X}{r_{\mathbb{R}}}}\ e^{-\frac{\nu_{\mathbb{C}}}{2}(\sqrt{x}|\sqrt{x})_{\mathbb{C}}}\ \frac{E^{\boldsymbol{m}}(\sqrt{x})}{E^{\boldsymbol{m}}_{\mathbb{R}}(e,e)} \\
&\quad = \int\limits_{\Omega} \frac{dx}{\Gamma_\Omega\left(\frac{d}{2r_{\mathbb{R}}}\right)}\ \Delta(x)^{\frac{d}{2r_{\mathbb{R}}}-\frac{d_X}{r_{\mathbb{R}}}}\ e^{-\nu_{\mathbb{R}}(x|e)_{\mathbb{R}}}\ \phi^{\boldsymbol{m}}_{\mathbb{R}}(x) \\
&\quad = \int\limits_{\Omega} \frac{dx}{\Gamma_\Omega\left(\frac{d}{2r_{\mathbb{R}}}\right)}\ \Delta(x)^{\frac{d}{2r_{\mathbb{R}}}-\frac{d_X}{r_{\mathbb{R}}}}\ e^{-\nu_{\mathbb{R}}(x|e)_{\mathbb{R}}}\ \Delta^{\boldsymbol{m}}(x) \\
&\quad = \int\limits_{\Omega} \frac{dx}{\Gamma_\Omega\left(\frac{d}{2r_{\mathbb{R}}}\right)}\ \Delta^{\boldsymbol{m}+\frac{d}{2r_{\mathbb{R}}}-\frac{d_X}{r_{\mathbb{R}}}}(x)\ e^{-(x|\nu_{\mathbb{R}}\,e)_{\mathbb{R}}} \\
&\quad = \frac{\Gamma_\Omega(\boldsymbol{m}+\frac{d}{2r_{\mathbb{R}}})}{\Gamma_\Omega(\frac{d}{2r_{\mathbb{R}}})}\ \Delta^{\boldsymbol{m}+\frac{d}{2r_{\mathbb{R}}}}\ (\nu_{\mathbb{R}}^{-1}\,e) = \left(\frac{d}{2r_{\mathbb{R}}}\right)_{\boldsymbol{m}}\ \nu_{\mathbb{R}}^{-|\boldsymbol{m}|-d/2}
\end{aligned}$$

since $\Delta^{\boldsymbol{m}+\frac{d}{2r_{\mathbb{R}}}}$ is homogeneous of total degree $|\boldsymbol{m}|+\frac{d}{2}$. Therefore

$$\frac{(\mathcal{B}_{\mathbb{R}}\,E^{\boldsymbol{m}})(0)}{E^{\boldsymbol{m}}_{\mathbb{R}}(e,e)} = \nu_{\mathbb{R}}^{d/2}\int\limits_{Z_{\mathbb{R}}} \frac{dw}{\pi^{d/2}}\ e^{-\frac{\nu_{\mathbb{C}}}{2}(w|w)_{\mathbb{C}}}\ \frac{E^{\boldsymbol{m}}(w)}{E^{\boldsymbol{m}}_{\mathbb{R}}(e,e)} = \left(\frac{d}{2r_{\mathbb{R}}}\right)_{\boldsymbol{m}}\ \nu_{\mathbb{R}}^{-|\boldsymbol{m}|}.$$

In the *non-compact* case, we have

$$h_{\mathbb{C}}(\sqrt{x},\sqrt{x}) = \Delta(e-x)^{r_{\mathbb{C}}/r_{\mathbb{R}}}$$

for $x \in \Omega \cap (e-\Omega)$. This yields

$$\begin{aligned}
&\int\limits_{Z_{\mathbb{R}}^-} \frac{dw}{\pi^{d/2}}\ h_{\mathbb{C}}(w,w)^{(\nu_{\mathbb{C}}-p_{\mathbb{C}})/2}\ \frac{E^{\boldsymbol{m}}(w)}{E^{\boldsymbol{m}}_{\mathbb{R}}(e,e)} \\
&\quad = \int\limits_{\Omega\cap(e-\Omega)} \frac{dx}{\Gamma_\Omega(\frac{d}{2r_{\mathbb{R}}})}\ \Delta(x)^{\frac{d}{2r_{\mathbb{R}}}-\frac{d_X}{r_{\mathbb{R}}}}\ h_{\mathbb{C}}(\sqrt{x},\sqrt{x})^{(\nu_{\mathbb{C}}-p_{\mathbb{C}})/2}\ \frac{E^{\boldsymbol{m}}(\sqrt{x})}{E^{\boldsymbol{m}}_{\mathbb{R}}(e,e)} \\
&\quad = \int\limits_{\Omega\cap(e-\Omega)} \frac{dx}{\Gamma_\Omega(\frac{d}{2r_{\mathbb{R}}})}\ \Delta(x)^{\frac{d}{2r_{\mathbb{R}}}-\frac{d_X}{r_{\mathbb{R}}}}\ \Delta(e-x)^{\nu_{\mathbb{R}}-p_{\mathbb{R}}}\ \phi^{\boldsymbol{m}}_{\mathbb{R}}(x) \\
&\quad = \int\limits_{\Omega\cap(e-\Omega)} \frac{dx}{\Gamma_\Omega(\frac{d}{2r_{\mathbb{R}}})}\ \Delta(x)^{\frac{d}{2r_{\mathbb{R}}}-\frac{d_X}{r_{\mathbb{R}}}}\ \Delta(e-x)^{\nu_{\mathbb{R}}-p_{\mathbb{R}}}\ \Delta^{\boldsymbol{m}}(x)
\end{aligned}$$

$$= \int\limits_{\Omega\cap(e-\Omega)} \frac{dx}{\Gamma_\Omega(\frac{d}{2r_\mathbb{R}})}\ \Delta^{\boldsymbol{m}+\frac{d}{2r_\mathbb{R}}-\frac{d_X}{r_\mathbb{R}}}(x)\ \Delta(e-x)^{\nu_\mathbb{R}-p_\mathbb{R}}$$

$$= \frac{\Gamma(\boldsymbol{m}+\frac{d}{2r_\mathbb{R}})\ \Gamma(\nu_\mathbb{R}-p_\mathbb{R}+\frac{d_X}{r_\mathbb{R}})}{\Gamma_\Omega\left(\frac{d}{2r_\mathbb{R}}\right)\ \Gamma(\nu_\mathbb{R}+\frac{d_X-d_Y}{2r_\mathbb{R}}+\boldsymbol{m})}$$

using the relation (2). Therefore

$$\frac{(\mathcal{B}^-_\mathbb{R}\, E^{\boldsymbol{m}})(0)}{E^{\boldsymbol{m}}_\mathbb{R}(e,e)} = \frac{\Gamma_\Omega\left(\nu_\mathbb{R}+\frac{d_X-d_Y}{2r_\mathbb{R}}\right)}{\Gamma_\Omega\left(\nu_\mathbb{R}-p_\mathbb{R}+\frac{d_X}{r_\mathbb{R}}\right)} \int\limits_{Z^-_\mathbb{R}} \frac{dw}{\pi^{d/2}}\ h_\mathbb{C}\,(w,w)^{(\nu_\mathbb{C}-p_\mathbb{C})/2}\ \frac{E^{\boldsymbol{m}}(w)}{E^{\boldsymbol{m}}_\mathbb{R}(e,e)}$$

$$= \frac{\Gamma_\Omega\left(\nu_\mathbb{R}+\frac{d_X-d_Y}{2r_\mathbb{R}}\right)}{\Gamma_\Omega\left(\nu_\mathbb{R}-p_\mathbb{R}+\frac{d_X}{r_\mathbb{R}}\right)}\ \frac{\Gamma(\boldsymbol{m}+\frac{d}{2r_\mathbb{R}})\ \Gamma(\nu_\mathbb{R}-p_\mathbb{R}+\frac{d_X}{r_\mathbb{R}})}{\Gamma_\Omega\left(\frac{d}{2r_\mathbb{R}}\right)\ \Gamma(\nu_\mathbb{R}+\frac{d_X-d_Y}{2r_\mathbb{R}}+\boldsymbol{m})} = \frac{(d/2r_\mathbb{R})_{\boldsymbol{m}}}{(\nu_\mathbb{R}+\frac{d_X-d_Y}{2r_\mathbb{R}})_{\boldsymbol{m}}}.$$

In the *compact* case, we have

$$h_\mathbb{C}\,(\sqrt{x},-\sqrt{x}) = \Delta(e+x)^{r_\mathbb{C}/r_\mathbb{R}}$$

for $x\in\Omega$ and obtain

$$\int\limits_{Z_\mathbb{R}} \frac{dw}{\pi^{d/2}}\ h_\mathbb{C}\,(w,-w)^{-(\nu_\mathbb{C}+p_\mathbb{C})/2}\ \frac{E^{\boldsymbol{m}}(w)}{E^{\boldsymbol{m}}_\mathbb{R}(e,e)}$$

$$= \int\limits_\Omega \frac{dx}{\Gamma_\Omega\left(\frac{d}{2r_\mathbb{R}}\right)}\ \Delta(x)^{\frac{d}{2r_\mathbb{R}}-\frac{d_X}{r_\mathbb{R}}}\ h_\mathbb{C}\,(\sqrt{x},-\sqrt{x})^{-(\nu_\mathbb{C}+p_\mathbb{C})/2}\ \frac{E^{\boldsymbol{m}}_\mathbb{R}(\sqrt{x})}{E^{\boldsymbol{m}}_\mathbb{R}(e,e)}$$

$$= \int\limits_\Omega \frac{dx}{\Gamma_\Omega\left(\frac{d}{2r_\mathbb{R}}\right)}\ \Delta(x)^{\frac{d}{2r_\mathbb{R}}-\frac{d_X}{r_\mathbb{R}}}\,\Delta(e+x)^{-\nu_\mathbb{R}-p_\mathbb{R}}\ \phi^{\boldsymbol{m}}_\mathbb{R}(x)$$

$$= \int\limits_\Omega \frac{dx}{\Gamma_\Omega\left(\frac{d}{2r_\mathbb{R}}\right)}\ \Delta(x)^{\frac{d}{2r_\mathbb{R}}-\frac{d_X}{r_\mathbb{R}}}\ \Delta(e+x)^{-\nu_\mathbb{R}-p_\mathbb{R}}\ \Delta^{\boldsymbol{m}}(x)$$

$$= \int\limits_\Omega \frac{dx}{\Gamma_\Omega\left(\frac{d}{2r_\mathbb{R}}\right)}\ \Delta^{\boldsymbol{m}+\frac{d}{2r_\mathbb{R}}-\frac{d_X}{r_\mathbb{R}}}(x)\ \Delta\,(e+x)^{-\nu_\mathbb{R}-p_\mathbb{R}}$$

$$= \frac{\Gamma_\Omega(\boldsymbol{m}+\frac{d}{2r_\mathbb{R}})\ \Gamma_\Omega(\nu_\mathbb{R}+p_\mathbb{R}-\boldsymbol{m}^*-\frac{d}{2r_\mathbb{R}})}{\Gamma_\Omega(\frac{d}{2r_\mathbb{R}})\ \Gamma_\Omega(\nu_\mathbb{R}+p_\mathbb{R})}.$$

Therefore

$$\frac{(\mathcal{B}^+_\mathbb{R}\, E^{\boldsymbol{m}})(0)}{E^{\boldsymbol{m}}_\mathbb{R}(e,e)} = \frac{\Gamma_\Omega\left(\nu_\mathbb{R}+p_\mathbb{R}\right)}{\Gamma_\Omega\left(\nu_\mathbb{R}+p_\mathbb{R}-\frac{d}{2r_\mathbb{R}}\right)} \int\limits_{Z_\mathbb{R}} \frac{dw}{\pi^{d/2}}\ h_\mathbb{C}\,(w,-w)^{-(\nu_\mathbb{C}+p_\mathbb{C})/2}\ \frac{E^{\boldsymbol{m}}(w)}{E^{\boldsymbol{m}}_\mathbb{R}(e,e)}$$

$$= \frac{\Gamma_\Omega\left(\nu_\mathbb{R}+p_\mathbb{R}\right)}{\Gamma_\Omega\left(\nu_\mathbb{R}+p_\mathbb{R}-\frac{d}{2r_\mathbb{R}}\right)}\ \frac{\Gamma_\Omega(\boldsymbol{m}+\frac{d}{2r_\mathbb{R}})\ \Gamma_\Omega(\nu_\mathbb{R}+p_\mathbb{R}-\boldsymbol{m}^*-\frac{d}{2r_\mathbb{R}})}{\Gamma_\Omega(\frac{d}{2r_\mathbb{R}})\ \Gamma_\Omega(\nu_\mathbb{R}+p_\mathbb{R})}$$

$$= \left(\frac{d}{2r_\mathbb{R}}\right)_{\boldsymbol{m}}\ \left(\nu_\mathbb{R}+p_\mathbb{R}-\frac{d}{2r_\mathbb{R}}\right)_{-\boldsymbol{m}^*}.$$

In all three cases, since $\mathcal{B}^\bullet_{\mathbb{R}}$ is a $G^\bullet_{\mathbb{R}}$-invariant operator on $Z^\bullet_{\mathbb{R}}$ there exist coefficients $c^\bullet_{\boldsymbol{m}}(\nu)$ $(\boldsymbol{m} \in \mathbb{N}^r_+)$ such that

$$(\mathcal{B}^\bullet_{\mathbb{R}} f)(0) = \sum_{\boldsymbol{m}} c^\bullet_{\boldsymbol{m}}(\nu_{\mathbb{R}})(\partial_{E^{\boldsymbol{m}}} f)(0)$$

for all functions f which are smooth near $0 \in Z_{\mathbb{R}}$. This implies

$$(\mathcal{B}^\bullet_{\mathbb{R}} E^{\boldsymbol{m}})(0) = \sum_{\boldsymbol{k}} c^\bullet_{\boldsymbol{k}} (\partial_{E^{\boldsymbol{k}}} E^{\boldsymbol{m}})(0) = c^\bullet_{\boldsymbol{m}} \|E^{\boldsymbol{m}}\|^2_{\mathbb{C}} = c^\bullet_{\boldsymbol{m}} E^{\boldsymbol{m}}_{\mathbb{R}}(e,e) \left(\frac{d}{2r_{\mathbb{R}}}\right)_{\boldsymbol{m}}.$$

It follows that

$$c^\bullet_{\boldsymbol{m}} = \frac{1}{(d/2r_{\mathbb{R}})_{\boldsymbol{m}}} \frac{(\mathcal{B}^\bullet_{\mathbb{R}} E^{\boldsymbol{m}})(0)}{E^{\boldsymbol{m}}_{\mathbb{R}}(e,e)}$$

has the values specified above. □

Remark 3.4. Since

$$\left(\nu_{\mathbb{R}} - p_{\mathbb{R}} + \frac{d}{2r_{\mathbb{R}}} + \frac{d_X}{r_{\mathbb{R}}}\right)_{\boldsymbol{m}} = \left(\nu_{\mathbb{R}} + \frac{d_X - d_Y}{2r_{\mathbb{R}}}\right)_{\boldsymbol{m}},$$

the non-compact case of the last theorem is in complete agreement with Theorem 14 of [EU1] (up to a factor of $(2r_{\mathbb{R}}/r_{\mathbb{C}})^{2|\boldsymbol{m}|}$, which is due to our different normalization of the inner product $(\cdot|\cdot)_{\mathbb{C}}$ with respect to $(\cdot|\cdot)_{\mathbb{R}}$.)

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On Nonlocal C^*-algebras of Two-dimensional Singular Integral Operators

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Abstract. Applying a local-trajectory method elaborated for studying nonlocal C^*-algebras associated with C^*-dynamical systems and the description of C^*-algebras generated by isometries, we construct a Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$ generated by the C^*-algebra $\mathfrak{A}$ of two-dimensional singular integral operators with continuous coefficients on a bounded closed simply connected domain $\overline{U} \subset \mathbb{R}^2$ with Liapunov boundary and by all unitary shift operators $W_g : f \mapsto J_g^{1/2}(f \circ g)$ where g runs a discrete amenable group G of quasiconformal diffeomorphisms of $\overline{U}$ onto itself with Hölder partial derivatives and the Jacobian J_g, and G acts on $\overline{U}$ topologically freely. As a result we establish Fredholm criteria for the operators $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$.

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1. Introduction

Given an arbitrary domain $U \subset \mathbb{R}^2$, let $\mathcal{B}(L^2(U))$ be the C^*-algebra of all bounded linear operators on the Hilbert space $L^2(U)$ with Lebesgue area measure, let $\mathcal{K} := \mathcal{K}(L^2(U))$ be the closed two-sided ideal of all compact operators in $\mathcal{B}(L^2(U))$, and let $\mathcal{B}(L^2(U))^\pi := \mathcal{B}(L^2(U))/\mathcal{K}$ denote the quotient C^*-algebra consisting of the cosets $A^\pi := A + \mathcal{K}$ with $A \in \mathcal{B}(L^2(U))$.

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Suppose now that U is a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ, G is a discrete amenable [12] group of quasiconformal diffeomorphisms $g : \overline{U} \to \overline{U}$ whose partial derivatives

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\right) \quad \text{and} \quad \frac{\partial g}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right) \tag{1.1}$$

satisfy a Hölder condition on $\overline{U}$, and the set Φ_g of all fixed points of g on $\overline{U}$ has empty interior for every shift $g \in G \setminus \{e\}$. With every $g \in G$ we associate a unitary weighted shift operator W_g on the Lebesgue space $L^2(U)$ given by

$$W_g f = J_g^{1/2}(f \circ g) \quad \text{for all } f \in L^2(U) \tag{1.2}$$

where $J_g(z) = \left|\frac{\partial g}{\partial z}\right|^2 - \left|\frac{\partial g}{\partial \overline{z}}\right|^2 > 0$ is the Jacobian of the quasiconformal diffeomorphism g (see [17]).

Given a domain $U \subset \mathbb{C}$, let S_U and S_U^* be the two-dimensional singular integral operators given by

$$(S_U f)(z) = -\frac{1}{\pi}\int_U \frac{f(w)}{(w-z)^2}\, dA(w), \quad (S_U^* f)(z) = -\frac{1}{\pi}\int_U \frac{f(w)}{(\overline{w}-\overline{z})^2}\, dA(w)$$

where $dA(z) = dxdy$ is the Lebesgue area measure. These operators are bounded on the space $L^2(U)$. We denote by

$$\mathfrak{A} := \text{alg}\left\{cI, S_U, S_U^* : c \in C(\overline{U})\right\} \tag{1.3}$$

the C^*-subalgebra of $\mathcal{B}(L^2(U))$ generated by all multiplication operators cI with $c \in C(\overline{U})$ and by the operators S_U and S_U^*.

The main goal of this paper is to study the Fredholmness of operators B (equivalently, the invertibility of cosets $B^\pi = B + \mathcal{K}$ [6]) in the nonlocal C^*-algebra

$$\mathfrak{B} := C^*(\mathfrak{A}, W_G) \subset \mathcal{B}(L^2(U)) \tag{1.4}$$

generated by all operators $A \in \mathfrak{A}$ and all shift operators W_g $(g \in G)$. By Lemma 2.6 in [15], which remains valid for arbitrary domains $U \subset \mathbb{C}$, the C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ contain the ideal $\mathcal{K}(L^2(U))$ of all compact operators in $\mathcal{B}(L^2(U))$.

In the present paper we construct Fredholm symbol calculi for the C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ and establish Fredholm criteria for the operators $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. To this end we apply the Allan-Douglas local principle [10], [6], the local-trajectory method elaborated in [13], [14], [3], Coburn's description of C^*-algebras generated by isometries [7]–[8], and the techniques of quasiconformal mappings. In studying the C^*-algebra $\mathfrak{A}$ we partially follow [24].

The paper is organized as follows. In Section 2 we describe the local-trajectory method, which is a non-local version of the Allan-Douglas local principle, for studying invertibility in nonlocal C^*-algebras associated with C^*-dynamical systems.

In Section 3, applying Coburn's results [7]–[8] on C^*-algebras generated by isometries, the orthogonal decomposition of the space $L^2(\Pi)$ over the complex upper half-plane $\Pi = \{z \in \mathbb{C} : \text{Im}\, z > 0\}$ in terms of true poly-Bergman and true anti-poly-Bergman spaces [25] and the characterization [3] of the operators S_Π and S_Π^* as nonunitary isometries on subspaces of $L^2(\Pi)$, we describe the spectrum of the C^*-algebra $\mathfrak{S} := \mathrm{alg}\,\{I, S_\Pi, S_\Pi^*\}$ of two-dimensional singular integral operators with constant coefficients on the space $L^2(\Pi)$ (cf. [24]).

In Section 4 we study the operators $W_\alpha S_U W_\alpha^{-1}$ and $W_\alpha S_U^* W_\alpha^{-1}$, where W_α is a unitary weighted shift operator for a quasiconformal diffeomorphism $\alpha : \overline{U} \to \overline{U}$.

In Section 5, applying the Allan-Douglas local principle and the results of Sections 3–4, we describe the spectrum of the quotient C^*-algebra $\mathfrak{A}^\pi = \mathfrak{A}/\mathcal{K}$ and establish a Fredholm criterion for the operators $A \in \mathfrak{A}$, where $\mathfrak{A}$ is given by (1.3).

Finally, in Section 6, making use of the local-trajectory method of Section 2 and the results of Section 5, we construct a Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$ given by (1.4) and obtain a Fredholm criterion for operators $B \in \mathfrak{B}$.

2. The local-trajectory method

2.1. Starting assumptions

Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{Z}$ a central C^*-subalgebra of $\mathcal{A}$ with the same unit I, G a discrete group with unit e, $U : g \mapsto U_g$ a homomorphism of the group G onto a group $U_G = \{U_g : g \in G\}$ of unitary elements such that $U_{g_1 g_2} = U_{g_1} U_{g_2}$ and $U_e = I$. Suppose $\mathcal{A}$ and U_G are contained in a C^*-algebra $\mathcal{D}$ and assume that

(A1) *for every $g \in G$ the mappings $\alpha_g : a \mapsto U_g\, a\, U_g^*$ are *-automorphisms of the C^*-algebras $\mathcal{A}$ and $\mathcal{Z}$;*

(A2) *G is an amenable discrete group.*

Amenable groups constitute a natural maximal class of groups for which one can establish an isomorphism of two C^*-algebras associated with C^*-dynamical systems (see, e.g., [2], [13], [14]). According to [12, § 1.2], a discrete group G is called *amenable* if the C^*-algebra $l^\infty(G)$ of all bounded complex-valued functions on G with sup-norm has an invariant mean, that is, a positive linear functional ρ of norm 1 (called a *state* [19]) satisfying the condition

$$\rho(f) = \rho({}_s f) = \rho(f_s) \ \text{ for all } s \in G \text{ and all } f \in l^\infty(G),$$

where $({}_s f)(g) = f(s^{-1}g)$, $(f_s)(g) = f(gs)$, $g \in G$. As is known (see, e.g., [1], [12], [14]), the class of amenable groups contains all finite groups, commutative groups, subexponential groups, and solvable groups. On the other hand, if a discrete group G contains the free discrete group F_2 with two generators, then G is not amenable.

Let $\mathcal{B} := C^*(\mathcal{A}, U_G)$ be the minimal C^*-algebra containing the unital C^*-algebra $\mathcal{A}$ and the group U_G. By virtue of (A1), $\mathcal{B}$ is the closure of the set $\mathcal{B}^0$ consisting of the elements $b = \sum a_g U_g$ where $a_g \in \mathcal{A}$ and g runs through finite subsets of G, and the algebraic operations are given as follows:

$$c\Big(\sum a_g U_g\Big) = \sum (c a_g) U_g \quad (c \in \mathbb{C}),$$

$$\sum a_g U_g + \sum a'_g U_g = \sum (a_g + a'_g) U_g \quad (a_g, a'_g \in \mathcal{A}),$$

$$\Big(\sum\nolimits_g a_g U_g\Big)\Big(\sum\nolimits_h a'_h U_h\Big) = \sum\nolimits_g \sum\nolimits_h (a_g \alpha_g(a'_h)) U_{gh}.$$

Let $M := M(\mathcal{Z})$ be the maximal ideal space of the (commutative) C^*-algebra $\mathcal{Z}$. By the Gelfand-Naimark theorem [20, § 16], $\mathcal{Z} \cong C(M)$ where $C(M)$ is the C^*-algebra of all continuous complex-valued functions on M. Under assumption (A1), identifying the non-zero multiplicative linear functionals φ_m of the algebra $\mathcal{Z}$ and the maximal ideals $m = \operatorname{Ker}\varphi_m \in M$, we obtain the homomorphism $g \mapsto \beta_g(\cdot)$ of the group G into the homeomorphism group of M according to the rule

$$z(\beta_g(m)) = (\alpha_g(z))(m), \quad z \in \mathcal{Z},\ m \in M,\ g \in G, \tag{2.1}$$

where $z(\cdot) \in C(M)$ is the Gelfand transform of the element $z \in \mathcal{Z}$. The set $G(m) := \{\beta_g(m) : g \in G\}$ is called the G-orbit of a point $m \in M$.

Let $P_{\mathcal{A}}$ be the set of all pure states on the C^*-algebra $\mathcal{A}$ equipped with the induced weak* topology, and let J_m denote the closed two-sided ideal of $\mathcal{A}$ generated by the maximal ideal $m \in M$ of the central C^*-algebra $\mathcal{Z} \subset \mathcal{A}$. By [5, Lemma 4.1], if $\mu \in P_{\mathcal{A}}$, then $\operatorname{Ker}\mu \supset J_m$ where $m := \mathcal{Z} \cap \operatorname{Ker}\mu \in M$, and therefore

$$P_{\mathcal{A}} = \bigcup\nolimits_{m \in M} \{\nu \in P_{\mathcal{A}} :\ \operatorname{Ker}\nu \supset J_m\}.$$

Let the following version of *topologically free* action of the group G hold (see [14], [3]):

(A3) *there is a set $M_0 \subset M$ such that for every finite set $G_0 \subset G \setminus \{e\}$ and every nonempty open set $V \subset P_{\mathcal{A}}$ there exists a state $\nu \in V$ such that $\beta_g(m_\nu) \neq m_\nu$ for all $g \in G_0$, where the point $m_\nu := \mathcal{Z} \cap \operatorname{Ker}\nu \in M$ belongs to the G-orbit $G(M_0) := \{\beta_g(m) :\ g \in G,\ m \in M_0\}$ of the set M_0.*

We say that the group G acts *freely* on M if the group $\{\beta_g :\ g \in G\}$ of homeomorphisms of M onto itself acts *freely* on M, that is, if $\beta_g(m) \neq m$ for all $g \in G \setminus \{e\}$ and all $m \in M$. Obviously, if the group G acts freely on M, then (A3) is fulfilled automatically.

If the C^*-algebra $\mathcal{A}$ is commutative, then the set $P_{\mathcal{A}}$ of all pure states of $\mathcal{A}$ coincides with the set of non-zero multiplicative linear functionals of $\mathcal{A}$ (see, e.g., [19, Theorem 5.1.6]). Therefore, choosing $\mathcal{Z} = \mathcal{A}$ and identifying the set of non-zero multiplicative linear functionals of $\mathcal{A}$ with the maximal ideal space $M(\mathcal{A})$ of $\mathcal{A}$, we can rewrite (A3) in the form

(A_0) *there is a set $M_0 \subset M(\mathcal{A})$ such that for every finite set $G_0 \subset G \setminus \{e\}$ and every nonempty open set $V \subset M(\mathcal{A})$ there exists a point $m_0 \in V \cap G(M_0)$ such that $\beta_g(m_0) \neq m_0$ for all $g \in G_0$.*

2.2. Trajectorial localization

Let the unital C^*-algebras $\mathcal{Z}$, $\mathcal{A}$, and $\mathcal{B} = C^*(\mathcal{A}, U_G)$ satisfy all the conditions of Subsection 2.1. In this subsection we recall an invertibility criterion for elements $b \in \mathcal{B}$ in terms of the invertibility of their local representatives associated with the G-orbits of points $m \in M$, where M is the compact space of maximal ideals of the central algebra $\mathcal{Z}$. As a result, we get a nonlocal version of the Allan-Douglas local principle (see [14], [3]).

For every $m \in M$, let J_m be the closed two-sided ideal of the algebra $\mathcal{A}$ generated by the maximal ideal m of the algebra $\mathcal{Z}$, and let $\mathcal{H}_m$ be the Hilbert space of an isometric representation $\widetilde{\pi}_m : \mathcal{A}/J_m \to \mathcal{B}(\mathcal{H}_m)$. We also consider the canonical *-homomorphism $\varrho_m : \mathcal{A} \to \mathcal{A}/J_m$ and the representation

$$\pi'_m : \mathcal{A} \to \mathcal{B}(\mathcal{H}_m), \quad a \mapsto (\widetilde{\pi}_m \circ \varrho_m)(a).$$

Since $\alpha_g(J_{\beta_g(m)}) = J_m$ for all $g \in G$ and all $m \in M$ in view of (A1), the quotient algebras $\mathcal{A}/J_{\beta_g(m)}$ and $\mathcal{A}/J_m$ are *-isomorphic. Then the spaces $\mathcal{H}_{\beta_g(m)}$ can be chosen equal for all $g \in G$.

Given $X \subset M$, let $\Omega(X)$ be the set of G-orbits of all points $m \in X$, let $H_\omega = \mathcal{H}_m$ where $m = m_\omega$ is an arbitrary fixed point of an orbit $\omega \in \Omega$ and $\Omega = \Omega(M)$, and let $l^2(G, H_\omega)$ be the Hilbert space of all functions $f : G \mapsto H_\omega$ such that $f(g) \neq 0$ for at most countable set of points $g \in G$ and $\sum \|f(g)\|^2_{H_\omega} < \infty$. For every $\omega \in \Omega$ we consider the representation $\pi_\omega : \mathcal{B} \to \mathcal{B}\big(l^2(G, H_\omega)\big)$ defined by

$$[\pi_\omega(a)f](g) = \pi'_m(\alpha_g(a))f(g), \quad [\pi_\omega(U_h)f](g) = f(gh)$$

for all $a \in \mathcal{A}$, all $g, h \in G$, and all $f \in l^2(G, H_\omega)$.

Theorem 2.1. [14, Theorem 4.1] *If assumptions* (A1)–(A3) *are satisfied, then an element $b \in \mathcal{B}$ is invertible* (*left invertible, right invertible*) *in $\mathcal{B}$ if and only if for every orbit $\omega \in \Omega$ the operator $\pi_\omega(b)$ is invertible* (*left invertible, right invertible*) *on the space $l^2(G, H_\omega)$ and, in the case of infinite Ω,*

$$\sup\left\{\left\|(\pi_\omega(b))^{-1}\right\| : \omega \in \Omega\right\} < \infty \tag{2.2}$$

(*resp.*, $\sup\left\{\left\|(\pi_\omega(b^*b))^{-1}\right\| : \omega \in \Omega\right\} < \infty$, $\sup\left\{\left\|(\pi_\omega(bb^*))^{-1}\right\| : \omega \in \Omega\right\} < \infty$).

By [14, Theorem 4.12], Theorem 2.1 is valid with Ω replaced by $\Omega_0 := \Omega(M_0)$.

The next result gives a sufficient condition that allows us to remove the uniform boundedness condition (2.2) for norms of inverse operators. Let $\overline{\omega}$ be the closure of an orbit $\omega \in \Omega$, and let ω' be the set of all limit points of ω.

Theorem 2.2. *If conditions* (A1)–(A3) *are satisfied, the C^*-algebra $\mathcal{Z}$ is separable, and $\bigcap_{m\in\omega} J_m = \bigcap_{m\in\overline{\omega}} J_m$ for every G-orbit $\omega \in \Omega$ such that $\overline{\omega} = \omega'$, then any element $b \in \mathcal{B}$ is invertible* (*left invertible, right invertible*) *in $\mathcal{B}$ if and only if for every orbit $\omega \in \Omega$ the operator $\pi_\omega(b)$ is invertible* (*left invertible, right invertible*) *on the space $l^2(G, H_\omega)$.*

Proof. By [14, Theorem 4.8], if (A1)–(A3) hold, the C^*-algebra $\mathcal{Z}$ is separable, and $\bigcap_{m\in\omega} J_m = \bigcap_{m\in\overline{\omega}} J_m$ for every G-orbit $\omega \in \Omega$ such that $\overline{\omega} = \omega'$, then for every irreducible representation π of the C^*-algebra $\mathcal{B}$ there exists a G-orbit $\omega \in \Omega$ possessing the property $\operatorname{Ker}\pi_\omega \subset \operatorname{Ker}\pi$. Then the required assertion follows from [14, Theorem 4.2]. □

Corollary 2.3. *If assumptions* (A1)–(A3) *are satisfied, $\mathcal{A} \cong \mathcal{Z}$ and the C^*-algebra $\mathcal{Z}$ is separable, then an element $b \in \mathcal{B}$ is invertible* (*left invertible, right invertible*) *in $\mathcal{B}$ if and only if for every orbit $\omega \in \Omega$ the operator $\pi_\omega(b)$ is invertible* (*left invertible, right invertible*) *on the space $l^2(G, H_\omega)$.*

Proof. Since the C^*-algebras $\mathcal{A}$ and $\mathcal{Z}$ are *-isomorphic, we can identify the closed two-sided ideals J_m of $\mathcal{A}$ and maximal ideals m of $\mathcal{Z}$. Identifying elements $z \in \mathcal{Z}$ and their Gelfand transforms $z(\cdot) \in C(M)$, we easily infer from the continuity of $z(\cdot)$ that any $z \in \mathcal{Z}$ belonging to all maximal ideals $m \in \omega$ also belongs to $\bigcap_{m\in\overline{\omega}} m$. Thus, $\bigcap_{m\in\omega} J_m = \bigcap_{m\in\overline{\omega}} J_m$, which in view of Theorem 2.2 completes the proof. □

3. C^*-algebra of two-dimensional singular integral operators with constant coefficients on the space $L^2(\Pi)$

Let $\Pi = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the open upper half-plane of the complex plain $\mathbb{C}$.

The poly-Bergman spaces $\mathcal{A}_n^2(\Pi)$ and the anti-poly-Bergman spaces $\widetilde{\mathcal{A}}_n^2(\Pi)$ are the Hilbert subspaces of $L^2(\Pi)$ that consist of n-differentiable functions such that, respectively, $(\partial/\partial\overline{z})^n f = 0$ and $(\partial/\partial z)^n f = 0$ (see, e.g., [11]). According to [25, Theorem 4.5], the space $L^2(\Pi)$ admits the following orthogonal decomposition:

$$L^2(\Pi) = \left(\bigoplus_{k=1}^{\infty} \mathcal{A}_{(k)}^2(\Pi) \right) \oplus \left(\bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}_{(k)}^2(\Pi) \right) \tag{3.1}$$

where the true poly-Bergman spaces of order n are defined as

$$\mathcal{A}_{(n)}^2(\Pi) = \mathcal{A}_n^2(\Pi) \cap \left[\mathcal{A}_{n-1}^2(\Pi)\right]^{\perp} \text{ for } n > 1, \ \mathcal{A}_{(1)}^2(\Pi) = \mathcal{A}_1^2(\Pi) := \mathcal{A}^2(\Pi),$$

and the true anti-poly-Bergman spaces of order n are defined by

$$\widetilde{\mathcal{A}}_{(n)}^2(\Pi) = \widetilde{\mathcal{A}}_n^2(\Pi) \cap \left[\widetilde{\mathcal{A}}_{n-1}^2(\Pi)\right]^{\perp} \text{ for } n > 1, \ \widetilde{\mathcal{A}}_{(1)}^2(\Pi) = \widetilde{\mathcal{A}}_1^2(\Pi) := \widetilde{\mathcal{A}}^2(\Pi).$$

The spaces $\mathcal{A}_{(k)}^2$, $\widetilde{\mathcal{A}}_{(k)}^2$ are related to the spaces $\mathcal{A}_{(k+1)}^2$, $\widetilde{\mathcal{A}}_{(k+1)}^2$ as follows.

Theorem 3.1. [16, Theorem 2.4] (also see [26]) *For every $k \in \mathbb{N}$, the operator S_Π is a unitary isomorphism of the space $\mathcal{A}_{(k)}^2$ onto $\mathcal{A}_{(k+1)}^2$ and of the space $\widetilde{\mathcal{A}}_{(k+1)}^2$ onto $\widetilde{\mathcal{A}}_{(k)}^2$, the operator S_Π^* is a unitary isomorphism of the space $\mathcal{A}_{(k+1)}^2$ onto $\mathcal{A}_{(k)}^2$ and of the space $\widetilde{\mathcal{A}}_{(k)}^2$ onto $\widetilde{\mathcal{A}}_{(k+1)}^2$, and $S_\Pi(\widetilde{\mathcal{A}}_{(1)}^2) = \{0\}$, $S_\Pi^*(\mathcal{A}_{(1)}^2) = \{0\}$.*

From (3.1) and Theorem 3.1 it follows that

$$L^2(\Pi) = \left(\bigoplus_{k=0}^{\infty} S_\Pi^k\big(\mathcal{A}^2(\Pi)\big) \right) \oplus \left(\bigoplus_{k=0}^{\infty} \left(S_\Pi^*\right)^k \big(\widetilde{\mathcal{A}}^2(\Pi)\big) \right).$$

Let $\mathfrak{S} = \operatorname{alg}\{I, S_\Pi, S_\Pi^*\}$ be the unital C^*-subalgebra of $\mathcal{B}(L^2(\Pi))$ generated by the identity operator I and the operators S_Π and S_Π^*, and let $\operatorname{Prim}\mathfrak{S}$ be the compact space of all primitive ideals (that is, kernels of non-zero irreducible representations) of $\mathfrak{S}$, which is equipped with the Jacobson topology. Consider the set $\widehat{\mathfrak{S}}$ of all unitary equivalence classes of non-zero irreducible representations of $\mathfrak{S}$ in Hilbert spaces. If (H, φ) is a non-zero irreducible representation φ of $\mathfrak{S}$ in a Hilbert space H, then $[H, \varphi]$ denotes its equivalence class in $\widehat{\mathfrak{S}}$. As is known (see, e.g., [19, Section 5.4]), $\widehat{\mathfrak{S}}$ becomes a compact topological space if it is endowed with the weakest topology making the canonical surjective map

$$\theta : \widehat{\mathfrak{S}} \to \operatorname{Prim} \mathfrak{S}, \quad [\mathrm{H}, \varphi] \mapsto \ker \varphi$$

continuous. The space $\widehat{\mathfrak{S}}$ is called the spectrum of the C^*-algebra $\mathfrak{S}$.

Let H^2 be the Hardy space of all complex-valued analytic functions defined on the open unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$ and equipped with the norm

$$\|f\|_{H^2} = \left(\sup_{r \in (0,1)} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{1/2}.$$

Consider the orthogonal projection $P = (I + S_{\mathbb{T}})/2$ on the Lebesgue space $L^2(\mathbb{T})$ where I is the identity operator and $S_{\mathbb{T}}$ is the Cauchy singular integral operator on the unit circle $\mathbb{T} = \partial D$,

$$(S_{\mathbb{T}}\varphi)(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\{z \in \mathbb{T} : |z-t| \geq \varepsilon\}} \frac{f(\tau)}{\tau - t} \, d\tau, \quad t \in \mathbb{T}.$$

We identify the Hardy space H^2 with the subspace $H^2(\mathbb{T}) := PL^2(\mathbb{T})$ of $L^2(\mathbb{T})$ consisting of the angular limits of all functions $f \in H^2$ on the unit circle $\mathbb{T}$. As is well known, the Toeplitz operators $T_a = PaP$ with symbols $a \in L^\infty(\mathbb{T})$ are bounded on the Hilbert space $H^2(\mathbb{T})$.

The next result was established in [24]. We give a more transparent proof.

Theorem 3.2. *The spectrum $\widehat{\mathfrak{S}}$ of the C^*-algebra $\mathfrak{S}$ can be parameterized by the points $z \in \mathbb{T} \cup \{\pm 2\}$ where the one-dimensional non-zero irreducible representations $\pi_z : \mathfrak{S} \to \mathbb{C}$ for every $z \in \mathbb{T}$ and the two infinite-dimensional non-zero irreducible representations $\pi_z : \mathfrak{S} \to \mathcal{B}(H^2(\mathbb{T}))$ for $z = \pm 2$ are given on the generators of the C^*-algebra $\mathfrak{S}$ by*

$$\pi_z(I) = 1, \quad \pi_z(S_\Pi) = z, \quad \pi_z(S_\Pi^*) = \overline{z} \quad \text{if } z \in \mathbb{T}, \tag{3.2}$$

$$\pi_z(I) = I, \quad \pi_z(S_\Pi) = T_z, \quad \pi_z(S_\Pi^*) = T_{\overline{z}} \quad \text{if } z = 2, \tag{3.3}$$

$$\pi_z(I) = I, \quad \pi_z(S_\Pi) = T_{\overline{z}}, \quad \pi_z(S_\Pi^*) = T_z \quad \text{if } z = -2, \tag{3.4}$$

where T_z and $T_{\overline{z}}$ are Toeplitz operators with symbols z and $\overline{z}$ on the space $H^2(\mathbb{T})$.

Proof. From Theorem 3.1 and (3.1) it follows that the mutually orthogonal subspaces $L^2_+(\Pi) := \bigoplus_{k=1}^{\infty} \mathcal{A}^2_{(k)}(\Pi)$ and $L^2_-(\Pi) := \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}^2_{(k)}(\Pi)$ of the Hilbert space $L^2(\Pi)$ are invariant under the action of the operators S_Π and S_Π^*, and the operator S_Π is a nonunitary isometry on the space $L^2_+(\Pi)$, while the operator S_Π^* is a nonunitary isometry on the space $L^2_-(\Pi)$. Hence, we infer from [7]–[8] that the spaces $\widehat{\mathfrak{S}}_\pm$ of all unitary equivalence classes of non-zero irreducible representations of the C^*-algebras $\mathfrak{S}_\pm := \{A|_{L^2_\pm(\Pi)} : A \in \mathfrak{S}\}$ consist of the classes parameterized, respectively, by $z \in \mathbb{T} \cup \{\pm 2\}$ and identified for $z \in \mathbb{T}$ with one-dimensional non-zero irreducible representations $\pi_z : \mathfrak{S} \to \mathbb{C}$ given by (3.2) and, for $z = \pm 2$, with infinite-dimensional non-zero irreducible representations $\pi_z : \mathfrak{S} \to \mathcal{B}(H^2(\mathbb{T}))$ given by (3.3)–(3.4). Combining these results for the C^*-algebras $\mathfrak{S}_\pm$ we complete the proof. □

4. Quasiconformal shifts and their applications

As is known (see, e.g., [17, Chapter 1]), a homeomorphism $\alpha = \alpha(z)$ of a domain $U \subset \mathbb{C}$ onto a domain $V \subset \mathbb{C}$ is called *quasiconformal* if α has locally integrable generalized derivatives $\frac{\partial\alpha}{\partial z}$ and $\frac{\partial\alpha}{\partial\overline{z}}$ of the form (1.1) that satisfy the inequality

$$\Big|\frac{\partial\alpha}{\partial\overline{z}}\Big| \le k\Big|\frac{\partial\alpha}{\partial z}\Big| \quad \text{where} \quad k = \text{const} < 1. \tag{4.1}$$

In particular, the partial derivatives (1.1) exist almost everywhere on U, α is differentiable almost everywhere, and the Jacobian $J_\alpha = \big|\frac{\partial\alpha}{\partial z}\big|^2 - \big|\frac{\partial\alpha}{\partial\overline{z}}\big|^2$ of the map $\alpha : U \to V$ is strictly positive for almost all $z \in U$ (see [17, Chapter 1, Subsection 9.4]).

Let now α be a quasiconformal diffeomorphism of a bounded closed domain $\overline{U} \subset \mathbb{C}$ onto itself. Since α is a differential bijection of the closed set $\overline{U}$, the Jacobian $J_\alpha > 0$ is separated from zero on $\overline{U}$. Hence, the operator $W_\alpha : f \mapsto \big|J_\alpha\big|^{1/2}(f \circ \alpha)$ is a unitary weighted shift operator on the Lebesgue space $L^2(U)$.

Substituting the shift α at points $w \in \overline{U}$ by its linear part

$$\widetilde{\alpha}_w(z) := \alpha(w) + \beta_w(z - w) + \gamma_w(\overline{z} - \overline{w})$$

where

$$\beta_w := \frac{\partial\alpha}{\partial z}(w), \quad \gamma_w := \frac{\partial\alpha}{\partial\overline{z}}(w) \tag{4.2}$$

and denoting the Jacobian J_α at points w by

$$J_w := |\beta_w|^2 - |\gamma_w|^2, \tag{4.3}$$

we established the following result jointly with L. Pessoa (see [15, Lemma 6.1]).

Lemma 4.1. *If α is a quasiconformal diffeomorphism of the closed unit disk $\overline{D}$ onto itself and its partial derivatives* (4.2) *satisfy a Hölder condition in $\overline{D}$, then the operators*

$$\begin{aligned} &W_\alpha S_D W_\alpha^{-1} - \frac{J_w}{\beta_w^2}\sum_{n=1}^{\infty}\left(\frac{\gamma_w}{\beta_w}\right)^{n-1}\big(S_D\big)^n + \frac{\overline{\gamma}_w}{\beta_w}I, \\ &W_\alpha S_D^* W_\alpha^{-1} - \frac{J_w}{\overline{\beta}_w^2}\sum_{n=1}^{\infty}\left(\frac{\overline{\gamma_w}}{\overline{\beta}_w}\right)^{n-1}\big(S_D^*\big)^n + \frac{\gamma_w}{\overline{\beta}_w}I \end{aligned} \tag{4.4}$$

are compact on the space $L^2(D)$.

Let U be a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ parameterized by a differentiable function $f : \mathbb{T} \to \Gamma$ with a Hölder derivative $f' \in H_\mu(\mathbb{T})$ $(\mu \in (0,1))$ separated from zero. Then, by the Kellogg-Warschawski theorem (see, e.g., [21, Theorem 3.6]), a conformal mapping φ of the open unit disc D onto U has a continuous extension φ to $\overline{D}$ and there is a positive constant $M < \infty$ such that

$$|\varphi'(z_1) - \varphi'(z_2)| \le M|z_1 - z_2|^\mu \quad \text{for all } \ z_1, z_2 \in \overline{D}.$$

Analyzing the proof of Lemma 4.1 we immediately derive the following corollary from that lemma.

Corollary 4.2. *If U is a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ and φ is a conformal mapping of the open unit disk D onto U, then the operators*

$$W_\varphi S_U W_\varphi^{-1} - (\overline{\varphi'}/\varphi')\, S_D, \quad W_\varphi S_U^* W_\varphi^{-1} - (\varphi'/\overline{\varphi'})\, S_D^*$$

are compact on the space $L^2(D)$.

Remark 4.3. *Corollary* 4.2 *remains valid with D replaced by another bounded simply connected domain V with Liapunov boundary.*

Representing a quasiconformal diffeomorphism $\alpha : \overline{U} \to \overline{U}$ in the form $\alpha = \varphi \circ \widetilde{\alpha} \circ \varphi^{-1}$ where $\widetilde{\alpha}$ is a quasiconformal diffeomorphism of $\overline{D}$ onto itself, and applying Lemma 4.1 and Corollary 4.2, we get the next generalization of Lemma 4.1.

Lemma 4.4. *If U is a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ and α is a quasiconformal diffeomorphism of $\overline{U}$ onto itself with partial derivatives* (4.2) *satisfying a Hölder condition on $\overline{U}$, then the operators* (4.4) *with D replaced by U are compact on the space $L^2(U)$.*

Applying equality (4.3) and the compactness of the commutators $cS_U - S_U cI$ for $c \in C(\overline{U})$, we infer that

$$\begin{aligned}\left[\frac{J_w}{\beta_w^2}\sum_{n=1}^{\infty}\left(\frac{\gamma_w}{\beta_w}\right)^{n-1}(S_U)^n - \frac{\overline{\gamma}_w}{\beta_w}I\right]^\pi &= \left[\frac{J_w}{\beta_w\gamma_w}\sum_{n=0}^{\infty}\left(\frac{\gamma_w}{\beta_w}\right)^{n}(S_U)^n - \frac{\overline{\beta_w}}{\gamma_w}I\right]^\pi \\ = \left[\frac{J_w}{\gamma_w}\left(\beta_w I - \gamma_w S_U\right)^{-1} - \frac{\overline{\beta_w}}{\gamma_w}I\right]^\pi &= \left[\left(\overline{\beta_w}S_U - \overline{\gamma_w}I\right)\left(\beta_w I - \gamma_w S_U\right)^{-1}\right]^\pi. \end{aligned} \tag{4.5}$$

Analogously,

$$\left[\frac{J_w}{\overline{\beta}_w^2}\sum_{n=1}^{\infty}\left(\frac{\overline{\gamma_w}}{\overline{\beta}_w}\right)^{n-1}(S_U^*)^n - \frac{\gamma_w}{\overline{\beta}_w}I\right]^\pi = \left[\left(\beta_w S_U^* - \gamma_w I\right)\left(\overline{\beta_w}I - \overline{\gamma_w}S_U^*\right)^{-1}\right]^\pi. \tag{4.6}$$

5. C^*-algebra of two-dimensional singular integral operators with continuous coefficients on the space $L^2(U)$

5.1. The Allan-Douglas local principle and the C^*-algebra $\mathfrak{A}$

To study the Fredholmness of operators $A \in \mathfrak{A}$ we apply the Allan-Douglas local principle (see, e.g., [10, Theorem 7.47], [6, Theorem 1.34]).

Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{Z}$ a central C*-subalgebra of $\mathcal{A}$ containing the identity of $\mathcal{A}$. Let $M(\mathcal{Z})$ denote the maximal ideal space of $\mathcal{Z}$. With every

$x \in M(\mathcal{Z})$ we associate the closed two-sided ideal J_x of $\mathcal{A}$ generated by the ideal x of $\mathcal{Z}$. Consider the quotient C^*-algebra $\mathcal{A}_x := \mathcal{A}/J_x$ and the canonical projection $\pi_x : \mathcal{A} \to \mathcal{A}_x$. Below we need the next part of the Allan-Douglas local principle.

Theorem 5.1. *Let $\mathcal{A}$ be a unital C^*-algebra satisfying the conditions mentioned above. If $a \in \mathcal{A}$, then a is invertible (left invertible, right invertible) in $\mathcal{A}$ if and only if for every $x \in M(\mathcal{Z})$ the coset $a_x := \pi_x(a)$ is invertible (left invertible, right invertible) in $\mathcal{A}_x$.*

Given a bounded simply connected domain $U \subset \mathbb{C}$ with Liapunov boundary Γ, consider the C^*-subalgebra $\mathfrak{A} = \operatorname{alg}\{cI, S_U, S_U^* : c \in C(\overline{U})\}$ of $\mathcal{B}(L^2(U))$. The C^*-algebra $\mathfrak{A}$ contains the ideal $\mathcal{K} = \mathcal{K}(L^2(U))$ of compact operators.

According to [23] (also see [4, Section 8.2]), an operator $A \in \mathcal{B}(L^2(U))$ is called an *operator of local type* if the commutators $cA - AcI$ are compact for all $c \in C(\overline{U})$. Let Λ be the C^*-algebra of all operators of local type on the space $L^2(U)$, and let $\Lambda^\pi := \Lambda/\mathcal{K}$ be the corresponding quotient C^*-algebra. From [18, Chapter X, Theorem 7.1] it follows that the singular integral operators S_U and S_U^* are of local type, which implies the following.

Lemma 5.2. *For every $A \in \mathfrak{A}$ and every function $c \in C(\overline{U})$, the commutators $cA - AcI$ are compact on the space $L^2(U)$.*

By Lemma 5.2, all the operators in the C^*-algebra $\mathfrak{A}$ are of local type, and $\mathcal{Z}^\pi := \{cI + \mathcal{K} : c \in C(\overline{U})\}$ is a central subalgebra of the C^*-algebra $\mathfrak{A}^\pi := \mathfrak{A}/\mathcal{K}$. Obviously, $\mathcal{Z}^\pi \cong C(\overline{U})$, and therefore the maximal ideal space $M(\mathcal{Z}^\pi)$ of $\mathcal{Z}^\pi$ can be identified with $\overline{U}$. For every point $w \in \overline{U}$, let J_w^π and $\widetilde{J}_w^\pi$ denote the closed two-sided ideals of the C^*-algebras $\mathfrak{A}^\pi$ and Λ^π, respectively, generated by the maximal ideal

$$I_w^\pi := \{cI + \mathcal{K} : c \in C(\overline{U}),\ c(w) = 0\} \subset \mathcal{Z}^\pi. \tag{5.1}$$

By analogy with [4, Proposition 8.6] one can prove that the ideals J_w^π and $\widetilde{J}_w^\pi$ have the form

$$\begin{aligned} J_w^\pi &= \{(cA)^\pi : c \in C(\overline{U}),\ c(w) = 0,\ A \in \mathfrak{A}\}, \\ \widetilde{J}_w^\pi &= \{(cA)^\pi : c \in C(\overline{U}),\ c(w) = 0,\ A \in \Lambda\}. \end{aligned} \tag{5.2}$$

With every $w \in \overline{U}$ we associate the local C^*-algebras

$$\mathfrak{A}_w^\pi := \{A^\pi + J_w^\pi : A \in \mathfrak{A}\}, \quad \widetilde{\mathfrak{A}}_w^\pi := \{A^\pi + \widetilde{J}_w^\pi : A \in \mathfrak{A}\}. \tag{5.3}$$

Lemma 5.3. *For every $w \in \overline{U}$ the map*

$$\psi_w : \mathfrak{A}_w^\pi \to \widetilde{\mathfrak{A}}_w^\pi,\ \ A^\pi + J_w^\pi \mapsto A^\pi + \widetilde{J}_w^\pi \tag{5.4}$$

*is an isometric *-isomorphism of the C^*-algebra $\mathfrak{A}_w^\pi$ onto the C^*-algebra $\widetilde{\mathfrak{A}}_w^\pi$.*

Proof. Clearly, the map ψ_w given by (5.4) is a *-homomorphism of the C^*-algebra $\mathfrak{A}_w^\pi$ onto the C^*-algebra $\widetilde{\mathfrak{A}}_w^\pi$. Let us show that ψ_w is an injective homomorphism. Indeed, if $A^\pi \in \mathfrak{A}^\pi$ and $A^\pi + J_w^\pi \in \operatorname{Ker}\psi_w$, then (5.4) implies that $A^\pi \in \mathfrak{A}^\pi \cap \widetilde{J}_w^\pi$. Then from (5.2) it follows that $A^\pi = (cB)^\pi$ where $c \in C(\overline{U})$, $c(w) = 0$ and

$B \in \Lambda$. Clearly, there exists a sequence $\{z_n\} \subset C(\overline{U})$ such that $z_n(w) = 0$ and $\lim_{n\to\infty} \|c(1-z_n)I\|_{\mathcal{B}(L^2(U))} = 0$. Hence $A^\pi = \lim_{n\to\infty} (z_n cB)^\pi$ in $\mathfrak{A}^\pi$, where $(z_n cB)^\pi \in J_w^\pi$ because $(z_n I)^\pi \in I_w^\pi$ (see (5.1)) and $(cB)^\pi \in \mathfrak{A}^\pi$. Finally, since the ideal J_w^π is closed, $A^\pi = \lim_{n\to\infty} (z_n cB)^\pi \in J_w^\pi$, that is, $A_w^\pi = 0^\pi + J_w^\pi$, which means the injectivity of ψ_w. Then, by [9, Corollary 1.8.3], ψ_w is an isometric *-isomorphism of the C^*-algebra $\mathfrak{A}_w^\pi$ onto the C^*-algebra $\widetilde{\mathfrak{A}}_w^\pi$. □

Applying Theorem 5.1 to the C^*-algebra $\mathfrak{A}^\pi$, we obtain the following.

Theorem 5.4. *An operator $A \in \mathfrak{A}$ is Fredholm on the space $L^2(U)$ if and only if for every $w \in \overline{U}$ the coset $A_w^\pi := A^\pi + J_w^\pi$ is invertible in the local C^*-algebra $\mathfrak{A}_w^\pi$.*

5.2. Local algebras

Let us study the local algebras $\mathfrak{A}_w^\pi$ associated to the points $w \in \overline{U}$ (see (5.3)).

If two C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are (isometrically) *-isomorphic, we will write $\mathcal{A}_1 \cong \mathcal{A}_2$. From the lemma below we can see that for $z \in \overline{U}$ there are two different types of local C^*-algebras.

Theorem 5.5. *Let U be a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ. Then for the C^*-algebra $\mathfrak{A} = \operatorname{alg}\{cI, S_U, S_U^* : c \in C(\overline{U})\}$ the following assertions hold:*

(i) *if $w \in U$, then $\mathfrak{A}_w^\pi \cong \operatorname{alg}\{I, S_{\mathbb{R}^2}, S_{\mathbb{R}^2}^*\}$ where the *-isomorphism is given by*

$$(cI)_w^\pi \mapsto c(w)I, \quad (S_U)_w^\pi \mapsto S_{\mathbb{R}^2}, \quad (S_U^*)_w^\pi \mapsto S_{\mathbb{R}^2}^*; \tag{5.5}$$

(ii) *if $w \in \Gamma$, then $\mathfrak{A}_w^\pi \cong \mathfrak{S}$ where the *-isomorphism is given by*

$$(cI)_w^\pi \mapsto c(w)I, \quad (S_U)_w^\pi \mapsto S_\Pi, \quad (S_U^*)_w^\pi \mapsto S_\Pi^*. \tag{5.6}$$

Proof. (i) Fix $w \in U$ and for $k > 0$ define the conformal mappings $\varphi_k : \mathbb{R}^2 \to \mathbb{R}^2$, $z \mapsto w + k(z - w)$. By Lemma 5.3, $\mathfrak{A}_w^\pi \cong \widetilde{\mathfrak{A}}_w^\pi$, where the *-isomorphism $\psi_w : \mathfrak{A}_w^\pi \to \widetilde{\mathfrak{A}}_w^\pi$ is given by (5.4). Considering the C^*-algebra $\mathfrak{A}$ as a C^*-subalgebra of $\mathcal{B}(L^2(\mathbb{R}^2))$ generated by the operators $(c\chi_U)I$, $\chi_U S_{\mathbb{R}^2}\chi_U I$ and $\chi_U S_{\mathbb{R}^2}^*\chi_U I$ where χ_U is the characteristic function of U, and identifying the closed two-sided ideals $\widetilde{J}_w^\pi$ in the C^*-algebras Λ^π and $\Lambda_{\mathbb{R}^2}^\pi$ where

$$\Lambda_{\mathbb{R}^2} := \Big\{A \in \mathcal{B}(L^2(\mathbb{R}^2)) : cA - AcI \in \mathcal{K}(L^2(\mathbb{R}^2)) \text{ for all } c \in C(\mathbb{R}^2 \cup \{\infty\})\Big\},$$

we infer from the property $((\chi_U - 1)I)^\pi \in \widetilde{J}_w^\pi$ that

$$\psi_w[(cI)_w^\pi] = (c(w)I)^\pi + \widetilde{J}_w^\pi, \quad \psi_w[(S_U)_w^\pi] = (S_{\mathbb{R}^2})^\pi + \widetilde{J}_w^\pi, \quad \psi_w[(S_U^*)_w^\pi] = (S_{\mathbb{R}^2}^*)^\pi + \widetilde{J}_w^\pi.$$

Hence, to any coset $A_w^\pi = \sum_i \prod_j \big[a_{i,j}S_U + b_{i,j}I + c_{i,j}S_U^*\big]_w^\pi \in \mathfrak{A}_w^\pi$ with $a_{i,j}, b_{i,j}, c_{i,j} \in C(\overline{U})$ we assign the coset $\psi_w[A_w^\pi] \in \widetilde{\mathfrak{A}}_w^\pi$ of the form $\widetilde{A}_w^\pi := \widetilde{A}^\pi + \widetilde{J}_w^\pi$ where

$$\widetilde{A} := \sum_i \prod_j \big[(a_{i,j}(w)S_{\mathbb{R}^2} + b_{i,j}(w)I + c_{i,j}(w)S_{\mathbb{R}^2}^*\big]. \tag{5.7}$$

Clearly, for every operator $\widetilde{A}$ of the form (5.7),

$$W_{\varphi_k}\widetilde{A}W_{\varphi_k}^{-1} = \widetilde{A}. \tag{5.8}$$

On the other hand, by analogy with [15, Proposition 7.5], we infer that

$$\operatorname*{s-lim}_{k\to 0}\left(W_{\varphi_k} T W_{\varphi_k}^{-1}\right)=0 \tag{5.9}$$

for every $T\in\Lambda$ such that $T^\pi\in\widetilde{J}_w^\pi$ and, in particular, for every $K\in\mathcal{K}(L^2(\mathbb{R}^2))$. Hence, we deduce from (5.8) and (5.9) that, for all such T,

$$\begin{aligned}\|\widetilde{A}\|_{\mathcal{B}(L^2(\mathbb{R}^2))} &\le \liminf_{k\to 0}\|W_{\varphi_k}\big(\widetilde{A}+T\big)W_{\varphi_k}^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^2))}\\ &= \|\widetilde{A}+W_{\varphi_k}TW_{\varphi_k}^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^2))},\end{aligned} \tag{5.10}$$

where the coset $\big[W_{\varphi_k}TW_{\varphi_k}^{-1}\big]^\pi$ belongs to the ideal $\widetilde{J}_w^\pi$ along with the coset T^π. Since for operators $\widetilde{A}$ of the form (5.7),

$$\|\widetilde{A}_w^\pi\| = \inf_{T\in\Lambda:\,T^\pi\in\widetilde{J}_w^\pi}\|\widetilde{A}+T\|_{\mathcal{B}(L^2(\mathbb{R}^2))}\le\|\widetilde{A}\|_{\mathcal{B}(L^2(\mathbb{R}^2))}, \tag{5.11}$$

we conclude from (5.10) and (5.11) that $\|\widetilde{A}_w^\pi\|=\|\widetilde{A}\|_{\mathcal{B}(L^2(\mathbb{R}^2))}$ for every $\widetilde{A}$ of the form (5.7), which implies that the C^*-algebras $\widetilde{\mathfrak{A}}_w^\pi=\psi_w[\mathfrak{A}_w^\pi]$ and $\operatorname{alg}\{I,S_{\mathbb{R}^2},S_{\mathbb{R}^2}^*\}$ are *-isomorphic. Thus, $\mathfrak{A}_w^\pi\cong\widetilde{\mathfrak{A}}_w^\pi\cong\operatorname{alg}\{I,S_{\mathbb{R}^2},S_{\mathbb{R}^2}^*\}$ where the *-isomorphism of the C^*-algebra $\mathfrak{A}_w^\pi$ onto the C^*-algebra $\operatorname{alg}\{I,S_{\mathbb{R}^2},S_{\mathbb{R}^2}^*\}$ is given by (5.5).

(ii) Let now $w\in\Gamma$. Consider a simply connected domain $V\subset\Pi$ with Liapunov boundary ∂V that contains a segment $[-1,1]\subset\mathbb{R}$. Then there exists a conformal map $\varphi:V\to U$ which admits a continuous extension to $\overline{V}$ with Hölder derivative φ' on $\overline{V}$ and such that $\varphi(0)=w$. Considering the C^*-algebra $\mathfrak{A}$ as a C^*-subalgebra of $\mathcal{B}(L^2(\mathbb{R}^2))$ according to part (i), and identifying the closed two-sided ideals $\widetilde{J}_w^\pi$ in the C^*-algebras Λ^π and $\Lambda_{\mathbb{R}^2}^\pi$, we infer that

$$\begin{gathered}\psi_w[(cI)_w^\pi]=(c(w)\chi_U I)^\pi+\widetilde{J}_w^\pi,\\ \psi_w[(S_U)_w^\pi]=(\chi_U S_{\mathbb{R}^2}\chi_U I)^\pi+\widetilde{J}_w^\pi,\quad \psi_w[(S_U^*)_w^\pi]=(\chi_U S_{\mathbb{R}^2}^*\chi_U I)^\pi+\widetilde{J}_w^\pi.\end{gathered} \tag{5.12}$$

By Corollary 4.2 and Remark 4.3,

$$\begin{aligned}W_\varphi\big[c(w)\chi_U I\big]^\pi W_\varphi^{-1}&=\big[c(w)\chi_V I\big]^\pi,\\ W_\varphi(S_U)^\pi W_\varphi^{-1}&=\big[(\overline{\varphi'}/\varphi')\,S_V\big]^\pi=\big[(\overline{\varphi'}/\varphi')\chi_V S_\Pi\chi_V I\big]^\pi,\\ W_\varphi(S_U^*)^\pi W_\varphi^{-1}&=\big[(\varphi'/\overline{\varphi'})\,S_V^*\big]^\pi=\big[(\varphi'/\overline{\varphi'})\chi_V S_\Pi^*\chi_V I\big]^\pi\end{aligned} \tag{5.13}$$

and $(W_\varphi)^\pi\widetilde{J}_w^\pi(W_\varphi^{-1})^\pi=\widetilde{J}_0^\pi$. Since $((\chi_V-\chi_\Pi)I)^\pi\in\widetilde{J}_0^\pi$, we conclude from (5.12) and (5.13) that

$$\begin{aligned}(W_\varphi)^\pi\psi_w[(cI)_w^\pi](W_\varphi^{-1})^\pi&=\big[c(w)\chi_\Pi I\big]_0^\pi,\\ (W_\varphi)^\pi\psi_w[(S_U)_w^\pi](W_\varphi^{-1})^\pi&=\big[(\overline{\varphi'(0)}/\varphi'(0))S_\Pi\big]_0^\pi,\\ (W_\varphi)^\pi\psi_w[(S_U^*)_w^\pi](W_\varphi^{-1})^\pi&=\big[(\varphi'(0)/\overline{\varphi'(0)})S_\Pi^*\big]_0^\pi.\end{aligned} \tag{5.14}$$

Hence, any coset $A_w^\pi=\sum_i\prod_j\big[a_{i,j}S_U+b_{i,j}I+c_{i,j}S_U^*\big]_w^\pi$ in the C^*-algebra $\mathfrak{A}_w^\pi$, where $a_{i,j},b_{i,j},c_{i,j}\in C(\overline{U})$, is transformed by (5.14) to the coset $\widehat{A}_0^\pi:=\widehat{A}^\pi+\widetilde{J}_0^\pi$

of the C^*-algebra $\mathfrak{S}_0^\pi$, where $\mathfrak{S} = \mathrm{alg}\,\{I, S_\Pi, S_\Pi^*\}$, $\mathfrak{S}_0^\pi = \{A^\pi + \widetilde{J}_0^\pi : A \in \mathfrak{S}\}$ and

$$\widehat{A} := \sum_i \prod_j \big[(a_{i,j}(w)(\overline{\varphi'(0)}/\varphi'(0))S_\Pi + b_{i,j}(w)I + c_{i,j}(w)(\varphi'(0)/\overline{\varphi'(0)})S_\Pi^*\big]. \tag{5.15}$$

Taking now an operator $T \in \Lambda$ such that $T^\pi \in \widetilde{J}_0^\pi$, applying the transforms $\widehat{A} + T \mapsto W_{\varphi_k}(\widehat{A} + T)W_{\varphi_k}^{-1}$ where $\varphi_k(z) = kz$ and passing to the strong limits $\operatorname*{s-lim}_{k\to 0} W_{\varphi_k}(\widehat{A} + T)W_{\varphi_k}^{-1}$, we infer by analogy with part (i) that, for operators $\widehat{A}$ of the form (5.15),

$$\|\widehat{A}_0^\pi\| = \inf_{T\in\Lambda:\, T^\pi\in\widetilde{J}_0^\pi} \|\widehat{A} + T\|_{\mathcal{B}(L^2(\mathbb{R}^2))} = \|\widehat{A}\|_{\mathcal{B}(L^2(\Pi))},$$

which due to (5.14) implies the *-isomorphism $\mathfrak{A}_w^\pi \cong \mathfrak{S}_0^\pi \cong \mathfrak{S}$ of the C^*-algebras $\mathfrak{A}_w^\pi$, $\mathfrak{S}_0^\pi$ and $\mathfrak{S}$, where the *-isomorphism $\mathfrak{A}_w^\pi \cong \mathfrak{S}$ is given on the generators of the C^*-algebra $\mathfrak{A}_w^\pi$ by

$$(cI)_w^\pi \mapsto c(w)I, \quad (S_U)_w^\pi \mapsto (\overline{\varphi'(0)}/\varphi'(0))S_\Pi, \quad (S_U)_w^\pi \mapsto (\varphi'(0)/\overline{\varphi'(0)})S_\Pi^*. \tag{5.16}$$

It remains to observe that $(\overline{\varphi'(0)}/\varphi'(0))S_\Pi$ is a nonunitary isometry on the space $L^2_+(\Pi)$ along with S_Π, and $(\varphi'(0)/\overline{\varphi'(0)})S_\Pi^*$ is a nonunitary isometry on the space $L^2_-(\Pi)$ along with S_Π^* because $|\overline{\varphi'(0)}/\varphi'(0)| = 1$. Since the C^*-algebra generated by any nonunitary isometry is *-isomorphic to the C^*-algebra generated by the unilateral shift of multiplicity one (see [7]), we infer from the proof of Theorem 3.2 that the map defined on the generators of the C^*-algebra $\mathfrak{S}$ by

$$I \mapsto I, \quad (\overline{\varphi'(0)}/\varphi'(0))S_\Pi \mapsto S_\Pi, \quad (\varphi'(0)/\overline{\varphi'(0)})S_\Pi^* \mapsto S_\Pi^* \tag{5.17}$$

is a *-isomorphism of the C^*-algebra $\mathfrak{S}$ onto itself. Finally, combining (5.16) and (5.17), we get the *-isomorphism of $\mathfrak{A}_w^\pi$ onto $\mathfrak{S}$ given by (5.6). □

From the formula for the Fourier transform of the kernels of multi-dimensional singular integral operators (see, e.g., [18, Chapter X, p. 249]) it follows that

$$S_{\mathbb{R}^2} = F^{-1}(\overline{\xi}/\xi)F, \quad S_{\mathbb{R}^2}^* = F^{-1}(\xi/\overline{\xi})F,$$

where F is the two-dimensional Fourier transform defined on $L^2(\mathbb{R}^2)$ by

$$(Fu)(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} u(t)e^{-ix\cdot t}dt, \quad x \in \mathbb{R}^2,$$

where $x \cdot t$ is the scalar product of vectors $x, t \in \mathbb{R}^2$, and F^{-1} is the inverse Fourier transform. Hence, $S_{\mathbb{R}^2}$ and $S_{\mathbb{R}^2}^*$ are unitary operators.

Remark 5.6. *Since the operators $S_{\mathbb{R}^2}$ and $S_{\mathbb{R}^2}^*$ are unitary on the space $L^2(\mathbb{R}^2)$, we conclude that the commutative C^*-algebra $\mathrm{alg}\,\{I, S_{\mathbb{R}^2}, S_{\mathbb{R}^2}^*\}$ is *-isomorphic to the C^*-algebra $C(\mathbb{T})$ where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the spectrum of $S_{\mathbb{R}^2}$ and the *-isomorphism is given by the Gelfand transform*

$$I \mapsto 1, \quad S_{\mathbb{R}^2} \mapsto z, \quad S_{\mathbb{R}^2}^* \mapsto \overline{z} \quad (z \in \mathbb{T}).$$

Combining Theorems 5.4, 5.5, Remark 5.6 and Theorem 3.2, we obtain the following result for the C^*-algebra $\mathfrak{A} = \mathrm{alg}\,\{cI, S_U, S_U^* : c \in C(\overline{U})\}$ (cf. [24]).

Theorem 5.7. *Let U be a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ. Then the spectrum $\widehat{\mathfrak{A}^\pi}$ of the quotient C^*-algebra $\mathfrak{A}^\pi$ can be parameterized by the points $(w,z) \in (\overline{U}\times\mathbb{T})\cup(\Gamma\times\{\pm 2\})$ where the one-dimensional non-zero irreducible representations $\pi_{w,z}:\mathfrak{A}^\pi\to\mathbb{C}$ for every $(w,z)\in\overline{U}\times\mathbb{T}$ and infinite-dimensional non-zero irreducible representations $\pi_{w,z}:\mathfrak{A}^\pi\to\mathcal{B}(H^2(\mathbb{T}))$ for every $(w,z)\in\Gamma\times\{\pm 2\}$ are given on the generators of the C^*-algebra $\mathfrak{A}^\pi$ by*

$$\begin{aligned}
&\pi_{w,z}([cI]^\pi)=c(w), \quad \pi_{w,z}([S_\Pi]^\pi)=z, \quad \pi_{w,z}([S_\Pi^*]^\pi)=\overline{z} \quad \text{if } (w,z)\in\overline{U}\times\mathbb{T},\\
&\pi_{w,z}([cI]^\pi)=c(w)I, \quad \pi_{w,z}([S_\Pi]^\pi)=T_z, \quad \pi_{w,z}([S_\Pi^*]^\pi)=T_{\overline{z}} \quad \text{if } (w,z)\in\Gamma\times\{2\},\\
&\pi_{w,z}([cI]^\pi)=c(w)I, \quad \pi_{w,z}([S_\Pi]^\pi)=T_{\overline{z}}, \quad \pi_{w,z}([S_\Pi^*]^\pi)=T_z \quad \text{if } (w,z)\in\Gamma\times\{-2\},
\end{aligned} \tag{5.18}$$

where T_z and $T_{\overline{z}}$ are Toeplitz operators with symbols z and $\overline{z}$ on the space $H^2(\mathbb{T})$.

Identifying numbers $b\in\mathbb{C}$ and the multiplication operators bI acting on the Hilbert space $H=\mathbb{C}$ and taking into account the continuity in view of (5.18) of the functions

$$\eta_A:\ \overline{U}\times\mathbb{T}\to\mathbb{C}, \quad (w,z)\mapsto\pi_{w,z}(A^\pi), \tag{5.19}$$

$$\eta_A^\pm:\ \Gamma\to\mathcal{B}(H^2(\mathbb{T})), \quad w\mapsto\pi_{w,\pm 2}(A^\pi) \tag{5.20}$$

for every $A\in\mathfrak{A}$, where $\pi_{w,z}(A^\pi)=\eta_A(w,z)$,

$$\begin{aligned}
\pi_{w,2}(A^\pi)&=\eta_A(w,T_z)=T_{\eta_A(w,z)}+K_1,\\
\pi_{w,-2}(A^\pi)&=\eta_A(w,T_{\overline{z}})=T_{\eta_A(w,\overline{z})}+K_2,
\end{aligned} \tag{5.21}$$

and K_1, K_2 are compact operators on the space $H^2(\mathbb{T})$, we immediately deduce the following result from Theorem 5.7.

Corollary 5.8. *Under the conditions of Theorem* 5.7*, an operator $A\in\mathfrak{A}$ is Fredholm on the space $L^2(U)$ if and only if for every $(w,z)\in(\overline{U}\times\mathbb{T})\cup(\Gamma\times\{\pm 2\})$ the operator $\pi_{w,z}(A^\pi)$ is invertible on the Hilbert space $H_{w,z}$, where $H_{w,z}=\mathbb{C}$ for $(w,z)\in\overline{U}\times\mathbb{T}$ and $H_{w,z}=H^2(\mathbb{T})$ for $(w,z)\in\Gamma\times\{\pm 2\}$.*

Thus, the operator function $\Psi(A):(w,z)\mapsto\pi_{w,z}(A^\pi)$ defined for $(w,z)\in(\overline{U}\times\mathbb{T})\cup(\Gamma\times\{\pm 2\})$ by (5.19)–(5.21) and equipped with the norm

$$\|\Psi(A)\|=\max\Big\{\max_{(w,z)\in\overline{U}\times\mathbb{T}}|\pi_{w,z}(A^\pi)|,\ \max_{(w,z)\in\Gamma\times\{\pm 2\}}\|\pi_{w,z}(A^\pi)\|_{\mathcal{B}(H^2(\mathbb{T}))}\Big\}$$

serves as a Fredholm symbol for operators $A\in\mathfrak{A}$, and the Fredholmness of $A\in\mathfrak{A}$ on the space $L^2(U)$ is equivalent to the invertibility of its Fredholm symbol $\Psi(A)$.

6. C^*-algebra of two-dimensional singular integral operators with shifts and continuous coefficients on the space $L^2(U)$

Let U be a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ, and let G be a discrete amenable group of quasiconformal diffeomorphisms

$g : \overline{U} \to \overline{U}$ whose partial derivatives $\frac{\partial g}{\partial z}$ and $\frac{\partial g}{\partial \overline{z}}$ satisfy a Hölder condition on $\overline{U}$. Suppose that for every $g \in G \setminus \{e\}$ the closed set $\Phi_g \subset \overline{U}$ of all fixed points of g has empty interior. To every $g \in G$ we assign the unitary weighted shift operator $W_g \in \mathcal{B}(L^2(U))$ given by (1.2). Consider the C^*-subalgebra $\mathfrak{B} := C^*(\mathfrak{A}, W_G)$ of $\mathcal{B}(L^2(U))$ generated by all operators $A \in \mathfrak{A}$ and all operators W_g with $g \in G$. Then $\mathfrak{B}$ contains the ideal $\mathcal{K} = \mathcal{K}(L^2(U))$, $\mathcal{Z}^\pi = \{cI + \mathcal{K} : c \in C(\overline{U})\}$ is a central subalgebra of the C^*-algebra $\mathfrak{A}^\pi = \mathfrak{A}/\mathcal{K}$, and $M(\mathcal{Z}^\pi) = C(\overline{U})$.

By Lemma 4.4, for every $g \in G$ the mappings $\alpha_g : A \mapsto W_g A W_g^*$ are *-automorphisms of the C^*-algebras $\mathfrak{A}^\pi$ and $\mathcal{Z}^\pi$. Thus, assumptions (A1) and (A2) of Subsection 2.1 are satisfied. The set $P_{\mathfrak{A}^\pi}$ of all pure states of the C^*-algebra $\mathfrak{A}^\pi$ consists of all functionals $\pi_{w,z}(A^\pi)$ for $(w,z) \in \overline{U} \times \mathbb{T}$ (these functionals simultaneously are one-dimensional representations of $\mathfrak{A}^\pi$) and all vector states $(\pi_{w,z}(A^\pi)\xi, \xi)$ for $(w,z) \in \Gamma \times \{\pm 2\}$ where $\xi \in H^2(\mathbb{T})$ are vectors of norm 1.

Since the interior of each set Φ_g $(g \in G \setminus \{e\})$ is empty, we easily infer from the continuity of the functions $\eta_A : \overline{U} \times \mathbb{T} \to \mathbb{C}$ $(A \in \mathfrak{A})$ that for every finite set $G_0 \subset G \setminus \{e\}$ and every open neighborhood $V_{w_0,z_0} \subset \overline{U} \times \mathbb{T}$ of any point $(w_0, z_0) \in \big(\bigcup_{g \in G_0} \Phi_g\big) \times \mathbb{T}$ there exists a point $(w, z_0) \in V_{w_0,z_0}$ such that $g(w) \neq w$ for all $g \in G_0$. In that case $M_0 = \overline{U}$, and for every $\varepsilon > 0$, every $(w_0, z_0) \in \big(\bigcup_{g \in G_0} \Phi_g\big) \times \mathbb{T}$ and every $A \in \mathfrak{A}$ there is a $\delta > 0$ such that

$$|\pi_{w,z_0}(A^\pi) - \pi_{w_0,z_0}(A^\pi)| = |\eta_A(w, z_0) - \eta_A(w_0, z_0)| < \varepsilon \quad \text{if } |w - w_0| < \delta.$$

On the other hand, from the continuity of functions $\eta_A^\pm : \Gamma \to \mathcal{B}(H^2(\mathbb{T}))$ $(A \in \mathfrak{A})$ it follows that for every finite set $G_0 \subset G \setminus \{e\}$ and every open neighborhood $V_{w_0} \subset \Gamma$ of any point $w_0 \in \bigcup_{g \in G_0} \Phi_g$ there exists a point $w \in V_{w_0}$ such that $g(w) \neq w$ for all $g \in G_0$. In that case again $M_0 = \overline{U}$, and for every $\varepsilon > 0$, every $w_0 \in \bigcup_{g \in G_0} \Phi_g$ and every $A \in \mathfrak{A}$ there is a $\delta > 0$ such that for any vector $\xi \in H^2(\mathbb{T})$ of norm 1,

$$\begin{aligned}
\big|(\pi_{w,2}(A^\pi)\xi, \xi) - (\pi_{w_0,2}(A^\pi)\xi, \xi)\big| &\le \big\|\pi_{w,2}(A^\pi) - \pi_{w_0,2}(A^\pi)\big\|_{\mathcal{B}(H^2(\mathbb{T}))} \\
&= \big\|\eta_A(w, T_z) - \eta_A(w_0, T_z)\big\|_{\mathcal{B}(H^2(\mathbb{T}))} < \varepsilon, \\
\big|(\pi_{w,-2}(A^\pi)\xi, \xi) - (\pi_{w_0,-2}(A^\pi)\xi, \xi)\big| &\le \big\|\pi_{w,-2}(A^\pi) - \pi_{w_0,-2}(A^\pi)\big\|_{\mathcal{B}(H^2(\mathbb{T}))} \\
&= \big\|\eta_A(w, T_{\overline{z}}) - \eta_A(w_0, T_{\overline{z}})\big\|_{\mathcal{B}(H^2(\mathbb{T}))} < \varepsilon
\end{aligned}$$

if $|w - w_0| < \delta$. Thus, condition (A3) is also fulfilled along with (A1)–(A2).

Hence, we can obtain a local-trajectory criterion for the invertibility of cosets $B^\pi \in \mathfrak{B}^\pi$ or, in other words, criterion for the Fredholmness of operators $B \in \mathfrak{B}$ on the basis of Theorems 2.1 and 2.2.

Since $\mathcal{Z}^\pi \cong C(\overline{U})$ and G is a discrete group of quasiconformal mappings of $\overline{U}$ onto itself, we conclude from (2.1) that $\beta_g = g$ for every $g \in G$. Then with every point $w \in \overline{U}$ we associate its G-orbit $G(w) = \{g(w) : g \in G\} \subset \overline{U}$.

For every $w \in \overline{U}$, consider the closed two-sided ideal J_w^π of the algebra $\mathfrak{A}^\pi$ generated by the maximal ideal I_w^π of the algebra $\mathcal{Z}^\pi \cong C(\overline{U})$, and let $\mathcal{H}_w$ be the Hilbert space of an isometric representation $\widetilde{\pi}_w : \mathfrak{A}^\pi/J_w^\pi \to \mathcal{B}(\mathcal{H}_w)$. We also consider the canonical *-homomorphism $\varrho_w : \mathfrak{A}^\pi \to \mathfrak{A}^\pi/J_w^\pi$ and the representation

$$\pi'_w : \mathfrak{A}^\pi \to \mathcal{B}(\mathcal{H}_w), \quad A^\pi \mapsto (\widetilde{\pi}_w \circ \varrho_w)(A^\pi).$$

Since $\alpha_g(J^\pi_{\beta_g(w)}) = J_w^\pi$ for all $g \in G$ and all $w \in \overline{U}$ in view of (A1), the quotient algebras $\mathfrak{A}^\pi/J^\pi_{g(w)}$ and $\mathfrak{A}^\pi/J^\pi_w$ are *-isomorphic. Then the spaces $\mathcal{H}_{g(w)}$ can be chosen equal for all $g \in G$.

According to Corollary 5.8, the descriptions of the Hilbert spaces $\mathcal{H}_w$ and representations π'_w for points $w \in U$ and points $w \in \Gamma$ are different. If $w \in U$, then $\mathcal{H}_w = \bigoplus_{z\in\mathbb{T}} \mathbb{C}$, while for $w \in \Gamma$ we have $\mathcal{H}_w = \big(\bigoplus_{z\in\mathbb{T}} \mathbb{C}\big) \oplus H^2(\mathbb{T}) \oplus H^2(\mathbb{T})$. Consequently, from Subsection 5.2 it follows that for $w \in U$ the representations $\pi'_w : \mathfrak{A}^\pi \to \mathcal{B}\big(\bigoplus_{z\in\mathbb{T}} \mathbb{C}\big)$ are given by $\pi'_w = \bigoplus_{z\in\mathbb{T}} \pi_{w,z}$. On the other hand, if $w \in \Gamma$, then the representations $\pi'_w : \mathfrak{A}^\pi \to \mathcal{B}\big(\big(\bigoplus_{z\in\mathbb{T}} \mathbb{C}\big) \oplus H^2(\mathbb{T}) \oplus H^2(\mathbb{T})\big)$ are given by $\pi'_w = \big(\bigoplus_{z\in\mathbb{T}} \pi_{w,z}\big) \oplus \pi_{w,2} \oplus \pi_{w,-2}$.

Let $\Omega := \Omega(\overline{U})$ be the set of all G-orbits of points $w \in \overline{U}$. Fix a point $t = t_\omega$ on every G-orbit $\omega \in \Omega$. Let $H_\omega = \mathcal{H}_t$ where $t = t_\omega$, and let $l^2(G, H_\omega)$ be the Hilbert space of all functions $f : G \mapsto H_\omega$ such that $f(g) \neq 0$ for at most countable set of points $g \in G$ and $\sum \|f(g)\|^2_{H_\omega} < \infty$. For every $\omega \in \Omega$ we consider the representation $\pi_\omega : \mathfrak{B} \to \mathcal{B}\big(l^2(G, H_\omega)\big)$ defined by

$$[\pi_\omega(A^\pi)f](g) = \pi'_t([\alpha_g(A)]^\pi)f(g), \quad [\pi_\omega([W_h]^\pi)f](g) = f(gh) \tag{6.1}$$

for all $A \in \mathfrak{A}$, all $g, h \in G$, and all $f \in l^2(G, H_\omega)$.

Taking $A = \sum_i \prod_j \big[a_{i,j} S_U + b_{i,j} I + c_{i,j} S_U^*\big] \in \mathfrak{B}$ and denoting $\beta_g(w) := \frac{\partial g}{\partial z}(w)$ and $\gamma_g(w) := \frac{\partial g}{\partial \overline{z}}(w)$, we infer from Lemma 4.4 and the equalities (4.5) and (4.6) that

$$\begin{aligned}
\big[\alpha_g(A)\big]^\pi_w &= \Big(W_g \sum\nolimits_i \prod\nolimits_j \big[a_{i,j} S_U + b_{i,j} I + c_{i,j} S_U^*\big] W_g^{-1}\Big)^\pi_w \\
&= \sum\nolimits_i \prod\nolimits_j \Big[a_{i,j}[g(w)]\big(\overline{\beta_g(w)} S_U - \overline{\gamma_g(w)} I\big)\big(\beta_g(w) I - \gamma_g(w) S_U\big)^{-1} \\
&\quad + b_{i,j}[g(w)] I + c_{i,j}[g(w)]\big(\beta_g(w) S_U^* - \gamma_g(w) I\big)\big(\overline{\beta_g(w)} I - \overline{\gamma_g(w)} S_U^*\big)^{-1}\Big]^\pi_w.
\end{aligned}$$

Hence, setting

$$\sigma_g(w, z) := \big(\overline{\beta_g(w)} - \overline{\gamma_g(w)}\, \overline{z}\big)\big(\beta_g(w) - \gamma_g(w) z\big)^{-1} z, \tag{6.2}$$

$$\sigma_g(w, T_z) := \big(\overline{\beta_g(w)} I - \overline{\gamma_g(w)} T_{\overline{z}}\big)\big(\beta_g(w) I - \gamma_g(w) T_z\big)^{-1} T_z, \tag{6.3}$$

we conclude from (6.3) that

$$[\sigma_g(w, T_z)]^* = \big(\beta_g(w) I - \gamma_g(w) T_z\big)\big(\overline{\beta_g(w)} I - \overline{\gamma_g(w)} T_{\overline{z}}\big)^{-1} T_{\overline{z}},$$

and therefore, taking into account (5.19) and (5.21), we obtain

$$\pi_{w,z}([\alpha_g(A)]^\pi) = \eta_A\big(g(w), \sigma_g(w,z)\big) \qquad \text{if } (w,z) \in \overline{U} \times \mathbb{T}, \tag{6.4}$$

$$\pi_{w,2}([\alpha_g(A)]^\pi) = \eta_A\big(g(w), \sigma_g(w,T_z)\big) \qquad \text{if } w \in \Gamma, \tag{6.5}$$

$$\pi_{w,-2}([\alpha_g(A)]^\pi) = \eta_A\big(g(w), [\sigma_g(w,T_z)]^*\big) \qquad \text{if } w \in \Gamma. \tag{6.6}$$

Thus every *-automorphism α_g of the C^*-algebra $\mathfrak{A}^\pi$ induces the homeomorphism λ_g of the compact $\overline{U} \times \mathbb{T}$ onto itself by the rule:

$$\lambda_g(w,z) = (g(w), \sigma_g(w,z)) \quad \text{for all } (w,z) \in \overline{U} \times \mathbb{T}.$$

It is easily seen that if G acts topologically freely on $M = \overline{U}$, then the group $\{\lambda_g : g \in G\}$ acts topologically freely on $\overline{U} \times \mathbb{T}$.

Setting $\Omega_U := \{G(w) : w \in U\}$, $\Omega_\Gamma := \{G(w) : w \in \Gamma\}$ and representing the spaces $l^2(G, H_\omega)$ (to within isometric isomorphisms) as $l^2(G,H_\omega) = \bigoplus_{z\in\mathbb{T}} l^2(G)$ if $\omega \in \Omega_U$, and

$$l^2(G,H_\omega) = \Big(\bigoplus\nolimits_{z\in\mathbb{T}} l^2(G)\Big) \oplus l^2(G, H^2(\mathbb{T})) \oplus l^2(G, H^2(\mathbb{T})) \quad \text{if } \omega \in \Omega_\Gamma,$$

we infer from (6.1) that the representation $\pi_\omega : \mathfrak{B}^\pi \to \mathcal{B}\big(l^2(G,H_\omega)\big)$ can be given by $\pi_\omega = \bigoplus_{z\in\mathbb{T}} \widetilde{\pi}_{\omega,z}$ if $\omega \subset U$, and

$$\pi_\omega = \Big(\bigoplus\nolimits_{z\in\mathbb{T}} \widetilde{\pi}_{\omega,z}\Big) \oplus \widetilde{\pi}_{\omega,2} \oplus \widetilde{\pi}_{\omega,-2} \quad \text{if } \omega \subset \Gamma,$$

where the representations $\widetilde{\pi}_{\omega,z} : \mathfrak{B}^\pi \to l^2(G)$ for $\omega \subset U$ and $z \in \mathbb{T}$ are defined for all $A \in \mathfrak{A}$, all $g, h \in G$ and all $f \in l^2(G)$ in view of (6.4) by

$$\begin{aligned} \big[\widetilde{\pi}_{\omega,z}(A^\pi)f\big](g) &= \pi_{w,z}([\alpha_g(A)]^\pi)f(g) = \eta_A\big(g(w), \sigma_g(w,z)\big)f(g), \\ \big[\widetilde{\pi}_{\omega,z}([W_h]^\pi)f\big](g) &= f(gh); \end{aligned} \tag{6.7}$$

and the representations $\widetilde{\pi}_{\omega,\pm 2} : \mathfrak{B}^\pi \to l^2(G, H^2(\mathbb{T}))$ for $\omega \subset \Gamma$ are defined for all $A \in \mathfrak{A}$, all $g, h \in G$ and all $f \in l^2(G, H^2(\mathbb{T}))$ in view of (6.5)–(6.6) by

$$\begin{aligned} \big[\widetilde{\pi}_{\omega,2}(A^\pi)f\big](g) &= \pi_{w,2}([\alpha_g(A)]^\pi)f(g) = \eta_A\big(g(w), \sigma_g(w,T_z)\big)f(g), \\ \big[\widetilde{\pi}_{\omega,-2}(A^\pi)f\big](g) &= \pi_{w,-2}([\alpha_g(A)]^\pi)f(g) = \eta_A\big(g(w), [\sigma_g(w,T_z)]^*\big)f(g), \\ \big[\widetilde{\pi}_{\omega,\pm 2}([W_h]^\pi)f\big](g) &= f(gh). \end{aligned} \tag{6.8}$$

Theorem 6.1. *Let U be a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ, let G be a discrete amenable group of quasiconformal diffeomorphisms $g : \overline{U} \to \overline{U}$ whose partial derivatives $\frac{\partial g}{\partial z}$ and $\frac{\partial g}{\partial \overline{z}}$ satisfy a Hölder condition on $\overline{U}$ and the sets Φ_g of fixed points for all $g \in G \setminus \{e\}$ have empty interiors, and let $\overline{\omega} \cap \mathbb{T} = \emptyset$ for every orbit $\omega \in \Omega_U$. Then an operator $B \in \mathfrak{B}$ is Fredholm (resp., n-normal, d-normal) on the space $L^2(U)$ if and only if for every $\omega \in \Omega_U$ and every $z \in \mathbb{T}$ the operators $\widetilde{\pi}_{\omega,z}(B^\pi)$ are invertible (resp., left invertible, right invertible) on the space $l^2(G)$ and for every $\omega \in \Omega_\Gamma$ the operators $\widetilde{\pi}_{\omega,\pm 2}(B^\pi)$ are invertible (resp., left invertible, right invertible) on the space $l^2(G, H^2(\mathbb{T}))$.*

Proof. Let us prove the Fredholm criterion for the operators $B \in \mathfrak{B}$ on the basis of Theorem 2.2 (the case of n- or d-normality of $B \in \mathfrak{B}$ is reduced to the Fredholmness of the operators B^*B or BB^*, respectively). Since the C^*-algebra $\mathfrak{B}$ satisfies conditions (A1)–(A3), it remains to check all other conditions of Theorem 2.2.

By the Stone-Weierstrass theorem (see, e.g., [22]), the C^*-algebra $C(\overline{U})$ of all continuous complex-valued functions on the compact $\overline{U}$ is the uniform closure of the set of all complex polynomials $P(z, \overline{z})$. Hence the C^*-algebra $C(\overline{U})$ is separable.

We will now prove that $\bigcap_{w\in\omega} J_w^\pi = \bigcap_{w\in\overline{\omega}} J_w^\pi$ for every G-orbit $\omega \in \Omega$ such that $\overline{\omega} = \omega'$. Let $\overline{\omega} = \omega'$ and $w \in \overline{\omega} \setminus \omega$. Then $w = \lim_{n\to\infty} t_n$ where $\{t_n\}$ is a sequence of points in the G-orbit ω. By [9, Theorem 2.9.7], every closed two-sided ideal J_w^π of the C^*-algebra $\mathfrak{A}^\pi$ is the intersection of the primitive ideals in $\mathfrak{A}^\pi$ (that is, kernels of non-zero irreducible representations of $\mathfrak{A}^\pi$ in Hilbert spaces) that contain J_w^π. Hence, $J_w^\pi = \bigcap_{z\in\mathbb{T}} \operatorname{Ker} \pi_{w,z}$ if $w \in U$, and

$$J_w^\pi = \Big(\bigcap\nolimits_{z\in\mathbb{T}} \operatorname{Ker} \pi_{w,z}\Big) \cap \operatorname{Ker} \pi_{w,2} \cap \operatorname{Ker} \pi_{w,-2} \quad \text{if } \ w \in \Gamma. \tag{6.9}$$

On the other hand, by [9, Proposition 2.4.9], for every pure state $\varrho = (\pi(\cdot)\xi, \xi)$ on a C^*-algebra $\mathfrak{A}^\pi$ where π is an irreducible representation of $\mathfrak{A}^\pi$ in a Hilbert space H and ξ is any vector in H of norm 1, it follows that $J_w^\pi \subset \operatorname{Ker} \varrho$ if and only if $J_w^\pi \subset \operatorname{Ker} \pi$. Let $\omega \in \Omega_U$. Then, because for every G-orbit $\omega \subset U$ the points $w \in \overline{\omega} \setminus \omega$ are in U, we conclude from the continuity of the function η_A on $\overline{U} \times \mathbb{T}$ that, for every state $\pi_{w,z}$ $(z \in \mathbb{T})$ and every coset $A^\pi \in \bigcap_{t\in\omega} J_t^\pi$,

$$\pi_{w,z}(A^\pi) = \eta_A(w, z) = \lim_{n\to\infty} \eta_A(t_n, z) = \lim_{n\to\infty} \pi_{t_n,z}(A^\pi) = 0.$$

Hence $A^\pi \in \bigcap_{z\in\mathbb{T}} \operatorname{Ker} \pi_{w,z} = J_w^\pi$, which implies that $\bigcap_{w\in\omega} J_w^\pi = \bigcap_{w\in\overline{\omega}} J_w^\pi$ for every G-orbit $\omega \in \Omega_U$.

Let now $\omega \subset \Omega_\Gamma$ and $A^\pi \in \bigcap_{t\in\omega} J_t^\pi$. Then by the part already proved we get $\pi_{w,z}(A^\pi) = 0$ for all $z \in \mathbb{T}$. Hence, for the pure states $\varrho_{w,\pm 2} := (\pi_{w,\pm 2}(\cdot)\xi, \xi)$ where $\xi \in H^2(\mathbb{T})$ are arbitrary vectors of norm 1, we infer by (5.2) and (5.21) that

$$\begin{aligned}
\varrho_{w,2}(A^\pi) &:= (\pi_{w,2}(A^\pi)\xi, \xi) = (\eta_A(w, T_z)\xi, \xi)\\
&= \lim_{n\to\infty} (\eta_A(t_n, T_z)\xi, \xi) = \lim_{n\to\infty} (\pi_{t_n,2}(A^\pi)\xi, \xi) = 0,\\
\varrho_{w,-2}(A^\pi) &:= (\pi_{w,-2}(A^\pi)\xi, \xi) = (\eta_A(w, T_{\overline{z}})\xi, \xi)\\
&= \lim_{n\to\infty} (\eta_A(t_n, T_{\overline{z}})\xi, \xi) = \lim_{n\to\infty} (\pi_{t_n,-2}(A^\pi)\xi, \xi) = 0.
\end{aligned}$$

Hence, in view of (5.21) and (6.9), we conclude that

$$A^\pi \in \Big(\bigcap\nolimits_{z\in\mathbb{T}} \operatorname{Ker} \pi_{w,z}\Big) \cap \operatorname{Ker} \pi_{w,2} \cap \operatorname{Ker} \pi_{w,-2} = J_w^\pi \quad \text{if } \ w \in \Gamma,$$

which implies that $\bigcap_{w\in\omega} J_w^\pi = \bigcap_{w\in\overline{\omega}} J_w^\pi$ for every G-orbit $\omega \in \Omega_\Gamma$ as well.

Since all the conditions of Theorem 2.2 are satisfied, we infer from this theorem that an operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(U)$ if and only if for every orbit $\omega \in \Omega$ the operator $\pi_\omega(B^\pi)$ is invertible on the space $l^2(G, H_\omega)$.

Let $\omega \in \Omega_U$. Then the invertibility of the operator $\pi_\omega(B^\pi)$ on the space $l^2(G, H_\omega)$ is equivalent to the invertibility of the operators $\widetilde{\pi}_{\omega,z}(B^\pi)$ on the space $l^2(G)$ for all $z \in \mathbb{T}$ with fulfillment of the condition $\sup\limits_{z\in\mathbb{T}} \big\|(\widetilde{\pi}_{\omega,z}(B^\pi))^{-1}\big\|_{\mathcal{B}(l^2(G))} < \infty$.

Fix $w \in \overline{U}$. By (4.1) with $k \in [0,1)$, for every $g \in G$ and every $z \in \mathbb{T}$ we get

$$|\beta_g(w) - \gamma_g(w)z| \ge |\beta_g(w)| - |\gamma_g(w)| \ge |\beta_g(w)|(1-k) \ge \min_{w\in\overline{U}} (J_w)^{1/2}(1-k) =: C > 0.$$

Consequently, for every $g \in G$ and all $z_1, z_2 \in \mathbb{T}$, we obtain the estimate

$$\left| \frac{\overline{\beta_g(w)} - \overline{\gamma_g(w)}\overline{z_1}}{\beta_g(w) - \gamma_g(w)z_1} - \frac{\overline{\beta_g(w)} - \overline{\gamma_g(w)}\overline{z_2}}{\beta_g(w) - \gamma_g(w)z_2} \right| \le C^{-2} \Big| |\gamma_g(w)|^2(\overline{z_1}z_2 - z_1\overline{z_2}) + 2i\mathrm{Im}\big[\overline{\beta_g(w)}\gamma_g(w)(z_1 - z_2)\big] \Big|. \tag{6.10}$$

Further, from (6.2) and (6.10) it follows in view of the uniform boundedness of $|\beta_g(w)|$ and $|\gamma_g(w)|$ on $\overline{U}$ that, for all $g \in G$ and given $w \in \overline{U}$, the functions $z \mapsto \sigma_g(w, z)$ are equicontinuous on $\mathbb{T}$. This implies that for every $A \in \mathfrak{A}$ and every $w \in \overline{U}$, the operator function

$$\mathbb{T} \to \mathcal{B}(l^2(G)), \quad z \mapsto \widetilde{\pi}_{\omega,z}(A^\pi) = \mathrm{diag}\big\{\eta_A\big(g(w), \sigma_g(w,z)\big)\big\}_{g\in G} I$$

is continuous on $\mathbb{T}$. Hence, for every $B \in \mathfrak{B}$ and every $\omega \in \Omega$ the operator-valued function $z \mapsto \widetilde{\pi}_{\omega,z}(B^\pi)$ also is continuous on $\mathbb{T}$, and therefore the condition $\sup\big\{\big\|(\widetilde{\pi}_{\omega,z}(B^\pi))^{-1}\big\|_{\mathcal{B}(l^2(G))} : z \in \mathbb{T}\big\} < \infty$ is fulfilled automatically for all $\omega \in \Omega$.

Let now $\omega \in \Omega_\Gamma$. Then the invertibility of the operator $\pi_\omega(B^\pi)$ on the space $l^2(G, H_\omega)$ is equivalent to the invertibility of the operators $\widetilde{\pi}_{\omega,z}(B^\pi)$ on the space $l^2(G)$ for all $z \in \mathbb{T}$, with the condition $\sup\big\{\big\|(\widetilde{\pi}_{\omega,z}(B^\pi))^{-1}\big\| : z \in \mathbb{T}\big\} < \infty$, and the invertibility of both the operators $\widetilde{\pi}_{\omega,\pm 2}(B^\pi)$ on the space $l^2(G, H^2(\mathbb{T}))$. But the uniform boundedness norms of operators $(\widetilde{\pi}_{\omega,z}(B^\pi))^{-1}$ with respect to $z \in \mathbb{T}$ for every $\omega \in \Omega_\Gamma$ was proved above.

Thus, the operators $\pi_\omega(B^\pi)$ are invertible on the spaces $l^2(G, H_\omega)$ for all $\omega \in \Omega$ if and only if for every $(\omega, z) \in \Omega_U \times \mathbb{T}$ the operators $\widetilde{\pi}_{\omega,z}(B^\pi)$ are invertible on the space $l^2(G)$ and for every $\omega \in \Omega_\Gamma$ the operators $\widetilde{\pi}_{\omega,\pm 2}(B^\pi)$ are invertible on the space $l^2(G, H^2(\mathbb{T}))$. □

Thus, the operator function $\widetilde{\Psi}(B) : (\omega, z) \mapsto \widetilde{\pi}_{\omega,z}(B^\pi)$ defined for $(\omega, z) \in (\Omega \times \mathbb{T}) \cup (\Omega_\Gamma \times \{\pm 2\})$ by (6.7)–(6.8) and equipped with the norm

$$\|\widetilde{\Psi}(B)\| = \max\Big\{ \max_{(\omega,z)\in\Omega\times\mathbb{T}} \|\widetilde{\pi}_{\omega,z}(B^\pi)\|_{\mathcal{B}(l^2(G))}, \max_{(\omega,z)\in\Omega_\Gamma\times\{\pm 2\}} \|\widetilde{\pi}_{\omega,z}(B^\pi)\|_{\mathcal{B}(l^2(G,H^2(\mathbb{T})))} \Big\}$$

serves as a Fredholm symbol for operators $B \in \mathfrak{B}$, and therefore Theorem 6.1 can be rewritten as follows.

Theorem 6.2. *Under the conditions of Theorem* 6.1*, an operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(U)$ if and only if its Fredholm symbol $\widetilde{\Psi}(B)$ is invertible.*

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The Riemann Boundary Value Problem on Non-rectifiable Curves and Fractal Dimensions

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Abstract. The aim of this work is to solve the Riemann boundary value problem on non-rectifiable curve. Its solvability depends on certain metric characteristics of the curve. We introduce new metric characteristics of dimensional type and new sharp conditions of solvability of the problem. In addition, we introduce and study a version of the Cauchy integral over non-rectifiable paths.

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Introduction

We consider the following boundary value problem for holomorphic functions. Let Γ be a closed Jordan curve on the complex plane $\mathbb{C}$ bounding finite domain D^+, and $D^- = \overline{\mathbb{C}} \setminus \overline{D^+}$. Find a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ such that $\Phi(\infty) = 0$, the boundary values $\lim_{D^+ \ni z \to t} \Phi(z) \equiv \Phi^+(t)$ and $\lim_{D^- \ni z \to t} \Phi(z) \equiv \Phi^-(t)$ exist for any $t \in \Gamma$, and

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), t \in \Gamma. \tag{0.1}$$

This boundary value problem is called the Riemann problem. It is well known and has numerous traditional applications in elasticity theory, hydro and aerodynamics and so on (see [1, 2]). Recently a number of authors explored its connections with theory of random matrices, non-classical estimates for orthogonal polynomials and so on (see, for instance, [3, 4]).

If $G(t) \equiv 1$, then the Riemann boundary value problem turns to so-called jump problem:

$$\Phi^+(t) - \Phi^-(t) = g(t), t \in \Gamma. \tag{0.2}$$

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The following classical result on this problem was obtained in XIX century by Sokhotskii, Plemelj and others (see, for instance, [1, 2]):

If the curve Γ *is piecewise-smooth and the jump* $g(t)$ *satisfies the Hölder condition*

$$\sup\left\{\frac{|f(t')-f(t'')|}{|t'-t''|^{\nu}} : t', t'' \in \Gamma, t' \neq t''\right\} \equiv h_{\nu}(f,\Gamma) < \infty \tag{0.3}$$

with exponent $\nu \in (0,1]$*, then unique solution of this problem is the Cauchy integral*

$$\Phi(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{g(\zeta)d\zeta}{\zeta - z}. \tag{0.4}$$

Below we denote $H_{\nu}(\Gamma)$ the set of all functions satisfying (0.3).

This result shows that solvability of the jump problem is closely connected with boundary properties of the Cauchy integral.

Solvation of the Riemann boundary value problem reduces to the jump problem by means of factorization. It will be discussed below.

During almost a century numerous authors studied continuity of boundary values of the Cauchy integral over non-smooth rectifiable curves. Finally, in 1979 E.M. Dynkin [5] and T. Salimov [6] published the following important result:

– *the Cauchy integral* (0.4) *over rectifiable curve* Γ *has boundary values* $\Phi^{\pm}$ *if* f *satisfies the Hölder condition with exponent*

$$\nu > \frac{1}{2}, \tag{0.5}$$

and this bound cannot be improved in the whole class of rectifiable curves.

This result implies that the jump problem on non-smooth rectifiable curves is solvable if the Hölder exponent of the jump exceeds $\frac{1}{2}$. This bound for the Hölder exponent cannot be improved on the whole class of rectifiable curves.

If curve Γ is not rectifiable, then customary definition of the Cauchy integral falls, but the Riemann boundary value problem and the jump problem keep sense and applicability.

In 1981 the author proved (see [7, 8]) solvability of the jump problem (0.2) on non-rectifiable closed curve Γ under assumption $g \in H_{\nu}(\Gamma)$,

$$\nu > \frac{1}{2}\,\mathrm{Dm}\,\Gamma, \tag{0.6}$$

where $\mathrm{Dm}\,\Gamma$ is the so-called box dimension (see [9]) or upper metric dimension (see [10]) of the curve Γ. It is defined by equality

$$\mathrm{Dm}\,\Gamma = \limsup_{\varepsilon\to 0}\frac{\log N(\varepsilon,\Gamma)}{-\log\varepsilon}, \tag{0.7}$$

where $N(\varepsilon,\Gamma)$ is the least number of disks of diameter ε covering the set Γ. As known, $1 \leq \mathrm{Dm}\,A \leq 2$ for any plane continuum A, and $\mathrm{Dm}\,\Gamma = 1$ for any rectifiable plane curve Γ. Therefore, if the curve Γ is rectifiable, then the condition (0.6) turns into the Dynkin–Salimov condition (0.5). But if the curve is not rectifiable, then the solution of jump problem is not representable by the Cauchy integral.

The condition (0.6) cannot be improved on the whole class of curves of fixed box dimension $d \in (1, 2)$, i.e., for any $\nu \in (0, d/2]$ there exist a closed Jordan curve Γ such that $\mathrm{Dm}\,\Gamma = d$ and a function $g \in H_\nu(\Gamma)$ such that the jump problem (0.2) has not solution (see [8]). But this condition does not sense certain features of the problem.

Example. Let $\Gamma = \partial D$, where $D = D_0 \bigcup(\bigcup_{j=1}^{\infty} R_j)$, $D_0 = \{z = x + iy : 0 < x < 1, -\phi(x) < y < 0\}$, $R_j = \{z = x + iy : x_j - \epsilon_j^p < x < x_j, 0 \le y < x_j\}$ for $j = 1, 2, \dots$, $\phi(x)$ is positive function with bounded variation, the positive sequences $\{x_j\}$, $\{\epsilon_j\}$ decrease to zero for $j \to \infty$, $x_j \le 1$, and the rectangles $R_j, j = 1, 2, \dots$ are disjoint. We shall see in the next section, that these sequences can be chosen so that $\mathrm{Dm}\,\Gamma$ equals to a fixed value $d \in (1, 2)$ for any $p > 1$, i.e., the condition (0.6) guarantees solvability of the jump problem for $\nu > d/2$. But for $p \to \infty$ the rectangles R_j turn into two-sided vertical cuts, and the jump problem on this curve turns the jump problem on rectifiable curve ∂D_0. The last problem is solvable for $\nu > 1/2$, what yields conjecture that for sufficiently large p the initial problem is solvable for certain $\nu \in (1/2, d/2)$. Below we shall prove this conjecture by means of new metric characteristics of non-rectifiable curves.

We consider these characteristics in Section 1. In Section 2 we represent solutions of the Riemann boundary value problem on closed non-rectifiable curve in terms of certain generalization of the Cauchy integral. In Section 3 we study this problem on open non-rectifiable arcs.

1. Approximation dimensions and integrations

The box dimension characterizes complexity of a non-rectifiable curve through its coverings. We apply more precious characterization through the rate of polygonal approximations of the curve in terms of its approximation dimension. It is introduced in the paper [11]. The rates of polygonal approximations of Γ from domains D^+ and D^- can differ. Therefore, here we introduce inner and outer approximation dimensions.

We say that a sequence of polygonal lines $G = \{\Gamma_1, \Gamma_2, \dots, \Gamma_n, \dots\}$ is inner (outer) polygonal approximation of the curve Γ if

1. $\Gamma_n = \partial P_n$, where P_n is open polygon or union of several open polygons (in the case of inner approximation all these polygons are finite, and in case of outer approximation one of them contains ∞), $n = 1, 2, \dots$;
2. $P_n \subset P_{n+1} \subset D^+$ (correspondingly, $P_n \subset P_{n+1} \subset D^-$) for any n;
3. $\lim_{n\to\infty} \mathrm{dist}(\Gamma_n, \Gamma) = 0$.

We put $\Delta_n = \overline{P_{n+1}} \setminus P_n$. This set is either closed polygon or union of several closed polygons, and some of them are multiply connected. Let λ_n stand for sum of perimeters of all connected components of Δ_n, and ω_n for diameter of the most disk contained in Δ_n. The sum

$$M_d(G) := \sum_{n=1}^{\infty} \lambda_n \omega_n^{d-1}$$

is called d-mass of polygonal approximation G.

Definition 1.1. Let $A^+(\Gamma)$ (correspondingly, $A^-(\Gamma)$) be set of all positive numbers d such that the curve Γ has inner inner (correspondingly, outer) polygonal approximation with finite d-mass. Then the values $\operatorname{Dma}^+\Gamma := \inf A^+(\Gamma)$ and $\operatorname{Dma}^-\Gamma := \inf A^-(\Gamma)$ are inner and outer approximation dimensions of this curve.

Let $\operatorname{Dma}\Gamma := \min(\operatorname{Dma}^+\Gamma, \operatorname{Dma}^-\Gamma)$, $\operatorname{Dma}^*\Gamma := \max(\operatorname{Dma}^+\Gamma, \operatorname{Dma}^-\Gamma)$. The value $\operatorname{Dma}\Gamma$ is introduced in the paper [11] as approximation dimension.

Theorem 1.2. *Any plane curve* Γ *satisfies inequalities*

$$1 \leq \operatorname{Dma}^{\pm}\Gamma \leq \operatorname{Dm}\Gamma \leq 2. \tag{1.1}$$

For any value $d \in (1,2)$ *there exist curves* $\Gamma_{1,2}$ *such that* $\operatorname{Dm}\Gamma = d$, $\operatorname{Dma}^+\Gamma_1 < d$ *and* $\operatorname{Dma}^-\Gamma_2 < d$.

The theorem shows that $\operatorname{Dma}^{\pm}\Gamma$ are characteristics of dimensional type, and, generally speaking, at least one of them is lesser than $\operatorname{Dm}\Gamma$. If Γ is rectifiable curve, then $\operatorname{Dma}^{\pm}\Gamma = 1$.

Proof. The inequality (1.1) can be proved in just the same way as the bound $1 \leq \operatorname{Dma}\Gamma \leq \operatorname{Dm}\Gamma \leq 2$ in the paper [11]. Then we construct the curves $\Gamma_{1,2}$ proving the last statement of the theorem. Let $\{a_k\}$ be a decreasing positive sequence such that $\sum_{k=1}^{\infty} a_k = 1$ and the series $\sum_{n=1}^{\infty} x_n$ diverge for $x_n = \sum_{k=n}^{\infty} a_k$. We consider vertical segments $\sigma_n := \{z = x_n + iy : 0 \leq y \leq x_n\}$ and evaluate box dimension of the set $\sigma := \overline{\cup_{n\geq 1}\sigma_n}$. Let us divide the plane into squares with side $\varepsilon > 0$ and denote by $N^{\diamond}(\varepsilon, \sigma)$ the number of squares intersecting σ. As known, $N(\varepsilon, A) \asymp N^{\diamond}(\varepsilon, A)$ for any compact set A, and we can replace N by $N^{\diamond}$ in the definition (0.7). We determine a number $n(\varepsilon)$ by relation $a_{n(\varepsilon)+1} \leq \varepsilon < a_{n(\varepsilon)}$. Then all segments with numbers $n \geq n(\varepsilon)$ are covered by N_1 squares filling the lower half of square $[0, x_{n(\varepsilon)}] \times [0, x_{n(\varepsilon)}]$ under its diagonal. Hence, $N_1 \asymp \varepsilon^{-2} x_{n(\varepsilon)}^2$. The rest segments $\sigma_k, k = 1, 2, \ldots, n(\varepsilon) - 1$, are covered by N_2 squares, and any square intersects only one segment. Whence, $N_2 \asymp \varepsilon^{-1} \sum_{k=1}^{n(\varepsilon)-1} x_k$ and

$$N^{\diamond}(\varepsilon, \sigma) \asymp \varepsilon^{-2} x_{n(\varepsilon)}^2 + \varepsilon^{-1} \sum_{k=1}^{n(\varepsilon)-1} x_k.$$

This relation enables us to evaluate $\operatorname{Dm}\sigma$ for a number of sequences $\{a_k\}$. Particularly, there is valid

Lemma 1.3. *If* $x_n \asymp \frac{1}{n^{\alpha}}$ *and* $a_n \asymp \frac{1}{n^{\alpha+1}}$ *for* $0 < \alpha < 1$, *then* $\operatorname{Dm}\sigma = \frac{2}{1+\alpha}$.

In what follows we put $x_n = \frac{1}{n^\alpha}$ for $\alpha = 2d^{-1} - 1, d \in (1,2)$. Then $\mathrm{Dm}\,\sigma = d$.

We fix $\beta > 1$ and consider rectangles $R_n := \{z = x + iy : x_n - a_n^\beta < x < x_n, 0 \leq y < x_n\}, n = 1, 2, \ldots$ Let $R := \bigcup_{n \geq 1} R_n$. We put $D_1^+ := \{z = x + iy : 0 < x < 1, -1 < y < 0\} \cup R$ (the square with a number of rectangular appendices), $D_2^+ := \{z = x + iy : 0 < x < 1, 0 < y < 1\} \setminus R$ (the square with a number of rectangular cuts), and $\Gamma_{1,2} = \partial D_{1,2}^+$. By virtue of Lemma 1.3 we have $\mathrm{Dm}\,\Gamma_1 = \mathrm{Dm}\,\Gamma_2 = d$. The curves Γ_1 and Γ_2 have evident inner and outer polygonal approximations with p-masses $M_p \asymp \sum_{n=1}^\infty x_n a_n^{\beta(p-1)}$. The series converges for $p > 1 + \frac{1-\alpha}{\beta(1+\alpha)} = 1 + \beta^{-1}(d-1)$. Thus, $\mathrm{Dma}^+\,\Gamma_1 \leq 1 + \frac{d-1}{\beta} < d, \mathrm{Dma}^-\,\Gamma_2 \leq 1 + \frac{d-1}{\beta} < d$. □

Now we consider a distributional approach to integration over closed non-rectifiable curves. Another approaches to this problem can be found in the works [12, 13, 14, 15] and similar one in [16]. We identify a function $F(z)$ on complex plane with distribution

$$\langle F, \varphi \rangle := \iint_{\mathbb{C}} F(z)\varphi(z) dz d\overline{z}, \varphi \in C_0^\infty(\mathbb{C}),$$

if the integral takes a sense. Let $F(z)$ be a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function such that the boundary values $\lim_{D^+ \ni z \to t} F(z) \equiv F^+(t)$, $\lim_{D^- \ni z \to t} F(z) \equiv F^-(t)$ exist for any $t \in \Gamma$, and $F(\infty) = 0$. We consider first a distribution $\overline{\partial} F$. It has support on the curve Γ. If the curve is rectifiable, then (see, for instance, [17])

$$\langle \overline{\partial} F, \varphi \rangle = \int_\Gamma (F^+(\zeta) - F^-(\zeta))\varphi(\zeta) d\zeta.$$

Thus, for non-rectifiable curve $\overline{\partial} F$ is a generalized integration with weight $F^+(\zeta) - F^-(\zeta)$. The integration without weight corresponds to functions F with unit jump on Γ. For instance, we can use to this end the characteristic function $\chi^+(z)$ of domain D^+, which equals to 1 in D^+ and to 0 in D^-. We call the distributions $\overline{\partial} F$ *primary integrations* and denote them $\int[F]$. We write $\int[F]\varphi d\zeta$ instead of $\langle \int[F], \varphi \rangle$. Obviously, $\overline{\partial} F$ vanishes on constants, and we can consider it as functional on factor $C^\infty(\mathbb{C})/\mathbb{C}$.

Let B be a finite domain such that $\Gamma \subset B$, $A = \overline{B}$. We denote $H^*(A, \nu) := \bigcup_{\mu \in (\nu, 1]} H_\mu(A)$. If we fix a sequence of exponents $\{\nu_j\}$ such that $1 > \nu_1 > \nu_2 > \cdots > \nu_j > \nu_{j+1} > \cdots$ and $\lim_{j \to \infty} \nu_j = \nu$, then the semi-norms $\{h_{\nu_j}(\cdot, A)\}$ turn $H^*(A, \nu)/\mathbf{C}$ into the Fréchet space. In what follows we write $H^*(A, \nu)$ instead of $H^*(A, \nu)/\mathbf{C}$ if this cannot cause ambiguity.

Theorem 1.4. *Let a holomorphic in* $\mathbf{C} \setminus \Gamma$ *function* F *be bounded on compact set* A *such that its interiority contains* Γ. *If* $\mathrm{Dma}^*\,\Gamma < 2$, *then primary integrations* $\int[F\chi^+]$, $\int[F\chi^-]$ *and* $\int[F]$ *are continuous in spaces* $H^*(A, \mathrm{Dma}^+\,\Gamma - 1)$, $H^*(A, \mathrm{Dma}^-\,\Gamma - 1)$ *and* $H^*(A, \mathrm{Dma}^*\,\Gamma - 1)$ *correspondingly. Here* $\chi^+(z)$ *and* $\chi^-(z)$ *are characteristic functions of domains* D^+ *and* D^-.

Proof. Let us prove continuity of the primary integration $\int[F\chi^+]$ in topology of the Fréchet space $H^*(A, \mathrm{Dma}^+\,\Gamma-1)$. We fix values d and ν such that $\mathrm{Dma}^+\,\Gamma < d < 2$, $1 > \nu > d-1$. By definition of the inner approximation dimension there exists a inner polygonal approximation G of the curve Γ such that $M_d(G) < \infty$. Let $G = \{\Gamma_1, \Gamma_2, \dots\}$. We put $\Gamma^* = \overline{\bigcup_{n\geq 1}\Gamma_n}$. Any function $\varphi \in C^\infty$ satisfies the Hölder condition with any exponent $\mu \leq 1$. We restrict φ on Γ^*, apply to this restriction the Whitney extension operator (see, for instance, [17]) and denote the obtained continuation φ^*. By virtue of well-known properties of the Whitney extension operator (see [17]) the function φ^* is defined in the whole complex plane, satisfies there the Hölder condition with any exponent $\mu \leq 1$ and equals to φ on the set Γ^*. In addition, it has partial derivatives of any order on $\mathbb{C}\setminus\Gamma^*$ and

$$|\nabla\varphi^*(z)| \leq Ch_\mu(\varphi, A)\,\mathrm{dist}^{\mu-1}(z, \Gamma^*);$$

here and below C stand for constants. Particularly, $|\nabla\varphi^*(z)| \leq Ch_1(\varphi, A)$, i.e., the first partial derivatives of φ^* are bounded. Consequently,

$$\begin{aligned}\int[F\chi^+]\varphi(\zeta)d\zeta &= \langle\overline{\partial}F\chi^+, \varphi\rangle = -\langle F\chi^+, \overline{\partial}\varphi\rangle = -\iint_{D^+} F(\zeta)\frac{\partial\varphi}{\partial\overline{\zeta}}d\zeta d\overline{\zeta}\\ &= -\sum_{n\geq 1}\iint_{\Delta_n} F(\zeta)\frac{\partial\varphi}{\partial\overline{\zeta}}d\zeta d\overline{\zeta} = \sum_{n\geq 1}\int_{\partial\Delta_n} F(\zeta)\varphi(\zeta)d\zeta\\ &= \sum_{n\geq 1}\int_{\partial\Delta_n} F(\zeta)\varphi^*(\zeta)d\zeta = -\sum_{n\geq 1}\iint_{\Delta_n} F(\zeta)\frac{\partial\varphi^*}{\partial\overline{\zeta}}d\zeta d\overline{\zeta}.\end{aligned}$$

Obviously, in polygonal domain Δ_n the function φ^* equals to the Whitney continuation of restriction of φ on $\partial\Delta_n$. Consequently, we can apply the following lemma from the paper [11].

Lemma 1.5. *Let δ be finite domain with rectifiable Jordan boundary γ, $f \in H_\nu(\gamma)$, and f^w is the Whitney continuation of f from γ. If $p < \frac{1}{1-\nu}$, then*

$$\iint_\delta |\nabla f^w|^p\,dxdy \leq Ch_\nu^p(f, \gamma)\lambda(\gamma)\omega^{1-p(1-\nu)}(\delta).$$

We fix a value p such that $d-1 = 1-p(1-\nu)$, i.e.,

$$p = \frac{2-d}{1-\nu}, \tag{1.2}$$

and obtain

$$\left|\int[F\chi^+]\varphi(\zeta)d\zeta\right| \leq CKS^{1/q}M_d^{1/p}(G)h_\nu(\varphi, A),$$

where S is area of D^+, $p^{-1}+q^{-1} = 1$ and K is upper bound for $|F|$. Thus, $\int[F\chi^+]$ is continuous in semi-norm $h_\nu(\cdot, A)$ for $\nu > \mathrm{Dma}^+\,\Gamma - 1$. Consequently, it is continuous in $H^*(A, \mathrm{Dma}^+\,\Gamma-1)$. The proof of continuity of $\int[F\chi^-]$ and $\int[F]$ in spaces $H^*(A, \mathrm{Dma}^-\,\Gamma-1)$ and $H^*(A, \mathrm{Dma}^*\,\Gamma-1)$ is analogous. □

As C^∞ is dense in $H^*(A,\nu)$, since the functionals $\int[F\chi^+]$, $\int[F\chi^-]$ and $\int[F]$ are continuable onto the spaces $H^*(A, \mathrm{Dma}^+\Gamma - 1)$, $H^*(A, \mathrm{Dma}^-\Gamma - 1)$ and $H^*(A, \mathrm{Dma}^*\Gamma - 1)$ correspondingly. The preceding proof gives us construction of these continuations. If a function f belongs to space $H^*(A, \mathrm{Dma}^+\Gamma - 1)$ (or $H^*(A, \mathrm{Dma}^-\Gamma - 1)$), then we fix an exponent $\nu > \mathrm{Dma}^+\Gamma - 1$ (correspondingly, $\nu > \mathrm{Dma}^-\Gamma - 1$) and a polygonal approximation G (inner or outer) of the curve Γ with finite d-mass such that $f \in H(A,\nu)$, $\nu > d - 1$ and $d > \mathrm{Dma}^+\Gamma$ (correspondingly, $d > \mathrm{Dma}^-\Gamma$), and put

$$\int[F\chi^\pm]f(\zeta)d\zeta = -\iint_{D^\pm} F(\zeta)\frac{\partial f^*}{\partial\overline{\zeta}}d\zeta d\overline{\zeta}, \tag{1.3}$$

where f^* is the Whitney extension of restriction of f on the set Γ^*. If we integrate over infinite domain D^-, then f^* must have compact support (for instance, we can multiply the Whitney extension by a smooth function with compact support equaling unit in a neighborhood of Γ). Obviously, $\int[F] = \int[F\chi^+] + \int[F\chi^-]$.

These functionals generate families of distributions

$$\left\langle \int[F]f, \varphi \right\rangle := \int[F]f(\zeta)\varphi(\zeta)d\zeta. \tag{1.4}$$

We call them *integrations* and write $\int[F]f\varphi d\zeta$ instead of $\langle\int[F]f,\varphi\rangle$.

2. The Cauchy transforms

Obviously, the supports of distributions $\int[F\chi^\pm]f$ and $\int[F]f$ belong to Γ. Therefore, we can apply them to the Cauchy kernel $\frac{1}{2\pi i(\zeta-z)}$ as function of variable ζ for $z \notin \Gamma$. Correctly speaking, we apply these distribution to a function $\omega_z(\zeta) \in C^\infty$ equaling to $\frac{1}{2\pi i(\zeta-z)}$ for $|\zeta - z| \geq \varepsilon$, where $0 < \varepsilon < \mathrm{dist}(z,\Gamma)$. As a result, we obtain the Cauchy transforms of integrations $\int[F\chi^\pm]f$ and $\int[F]f$. Let us denote

$$C[F]f(z) := \left\langle \int[F]f, \frac{1}{2\pi i(\zeta - z)} \right\rangle.$$

The representation (1.3) yields the following result.

Lemma 2.1. *The Cauchy transforms of integrations $\int[F\chi^\pm]f$ and $\int[F]f$ are representable as follows:*

1. *if $f \in H^*(A, \mathrm{Dma}^+\Gamma - 1)$, then*

$$C[F\chi^+]f(z) = F(z)\chi^+(z)f^*(z) - \frac{1}{2\pi i}\iint_{D^+}\frac{\partial f^*}{\partial\overline{\zeta}}\frac{F(\zeta)d\zeta d\overline{\zeta}}{\zeta - z},$$

where f^* is the Whitney extension of restriction of f on the set $\Gamma^* = \overline{\bigcup_{n\geq 1}\Gamma_n}$, and $G = \{\Gamma_n\}$ is inner polygonal approximation of Γ with finite d-mass, $\nu + 1 > d > \mathrm{Dma}^+\Gamma$, $f \in H_\nu(A)$;

2. *if* $f \in H^*(A, \mathrm{Dma}^- \Gamma - 1)$, *then*

$$C[F\chi^-]f(z) = F(z)\chi^-(z)f^*(z) - \frac{1}{2\pi i}\iint_{D^-} \frac{\partial f^*}{\partial \overline{\zeta}} \frac{F(\zeta)d\zeta d\overline{\zeta}}{\zeta - z},$$

where f^* *is the Whitney extension with compact support of restriction of* f *on the set* $\Gamma^* = \overline{\bigcup_{n\geq 1} \Gamma_n}$, *and* $G = \{\Gamma_n\}$ *is outer polygonal approximation of* Γ *with finite d-mass,* $\nu + 1 > d > \mathrm{Dma}^- \Gamma$, $f \in H_\nu(A)$*;*

3. *if* $f \in H^*(A, \mathrm{Dma}^* \Gamma - 1)$, *then*

$$C[F]f(z) = F(z)f^*(z) - \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\partial f^*}{\partial \overline{\zeta}} \frac{F(\zeta)d\zeta d\overline{\zeta}}{\zeta - z},$$

where f^* *is the Whitney extension with compact support of restriction of* f *on the union of sets* Γ^* *from the previous items.*

The last terms of these representations are well known integral operators (see, for instance, [17]). If $\frac{\partial f^*}{\partial \overline{\zeta}} \in L^p_{loc}$ for $p > 2$, then these terms represent functions, which are continuous in the whole complex plane and satisfy there the Hölder condition with exponent $1 - \frac{2}{p}$. The exponent p is determined by equality (1.2). It exceeds 2 for $\nu > d/2$. As a result, we obtain

Theorem 2.2. *Let function* $F(z)$ *be holomorphic in* $\overline{\mathbf{C}} \setminus \Gamma$, *continuous in* $\overline{D^+}$ *and* $\overline{D^-}$, *and* $F(\infty) = 0$. *Then the following propositions are valid:*

1. *if* $f \in H^*(A, \mathrm{Dma}^+ \Gamma/2)$, *then the function* $\Phi(z) := C[F\chi^+]f(z)$ *is continuous in* $\overline{D^+}$ *and* $\overline{D^-}$, *and*

$$\Phi^+(t) - \Phi^-(t) = F^+(t)f(t), t \in \Gamma;$$

in addition, if $f \in H_\nu(A)$ *and* F *satisfy in* $\overline{D^+}$ *and* $\overline{D^-}$ *the Hölder condition with exponent*

$$\mu(\nu, d) := \frac{2\nu - d}{2 - d}, \tag{2.1}$$

where $d = \mathrm{Dma}^+ \Gamma$, *then the restrictions of* Φ *on* $\overline{D^+}$ *and* $\overline{D^-}$ *satisfy the Hölder condition with any exponent lesser than* $\mu(\nu, d)$*;*

2. *if* $f \in H^*(A, \mathrm{Dma}^- \Gamma/2)$, *then the function* $\Phi(z) := C[F\chi^-]f(z)$ *is continuous in* $\overline{D^+}$ *and* $\overline{D^-}$, *and*

$$\Phi^+(t) - \Phi^-(t) = -F^-(t)f(t), t \in \Gamma;$$

in addition, if $f \in H_\nu(A)$ *and* F *satisfy in* $\overline{D^+}$ *and* $\overline{D^-}$ *the Hölder condition with exponent* $\mu(\nu, d)$, *where* $d = \mathrm{Dma}^- \Gamma$, *then the restrictions of* Φ *on* $\overline{D^+}$ *and* $\overline{D^-}$ *satisfy the Hölder condition with any exponent lesser than* $\mu(\nu, d)$*;*

3. *if* $f \in H^*(A, \mathrm{Dma}^* \Gamma/2)$, *then the function* $\Phi(z) := C[F]f(z)$ *is continuous in* $\overline{D^+}$ *and* $\overline{D^-}$, *and*

$$\Phi^+(t) - \Phi^-(t) = (F^+(t) - F^-(t))f(t), t \in \Gamma;$$

in addition, if $f \in H_\nu(A)$ *and* F *satisfy in* $\overline{D^+}$ *and* $\overline{D^-}$ *the Hölder condition* *with exponent* $\mu(\nu, d)$*, where* $d = \mathrm{Dma}^* \Gamma$*, then the restrictions of* Φ *on* $\overline{D^+}$ *and* $\overline{D^-}$ *satisfy the Hölder condition with any exponent lesser than* $\mu(\nu, d)$*.*

Thus, we obtain analog of the Sokhotskii–Plemelj formula for non-rectifiable curves, what suffices for solution of the jump problem.

Corollary 2.3. *If* $g \in H^*(\Gamma, \mathrm{Dma}\,\Gamma/2)$*, then the jump problem* (0.2) *is solvable, and a solution is equal to* $C[\chi^+]g^w(z)$ *for* $\mathrm{Dma}\,\Gamma = \mathrm{Dma}^+\,\Gamma$ *and to* $C[-\chi^-]g^w(z)$ *for* $\mathrm{Dma}\,\Gamma = \mathrm{Dma}^-\,\Gamma$*. Here* g^w *is the Whitney extension of function* g*.*

Proof. If $f \in H^*(\Gamma, \mathrm{Dma}\,\Gamma/2)$, then $f^w \in H^*(A, \mathrm{Dma}\,\Gamma/2)$ for any compact A containing Γ in its interiority, and propositions 1, 2 of Theorem 2.2 turn into the solvability conditions for the problem (0.2). □

The solvability of the jump problem under restriction $f \in H^*(\Gamma, \mathrm{Dma}\,\Gamma/2)$ is proved in the paper [11]. Here we add its representability by the Cauchy transform, which is generalization of the Cauchy integral for non-rectifiable curves.

The assumption $f \in H^*(\Gamma, \mathrm{Dma}\,\Gamma/2)$ is weaker than condition (0.6) by virtue of Theorem 1.2.

Proposition 3 of Theorem 2.2 enables us to prove another condition for solvability of the jump problem.

Corollary 2.4. *If* $g_1 \in H^*(\Gamma, \mathrm{Dma}^*\,\Gamma/2)$ *and* g_2 *is defined on* Γ *function such that jump problem* (0.2) *is solvable for* $g = g_2$*, then this problem is solvable for* $g = g_1 g_2$*.*

Proof. Let $F(z)$ be a solution of the jump problem (0.2) for $g = g_2$, i.e., $F^+(t) - F^-(t) = g_2(t)$ for $t \in \Gamma$. According proposition g of Theorem 2.2 the Cauchy transform $C[F]g_1(z)$ is a solution of the problem for $g = g_1 g_2$. □

A solution of the jump problem on non-rectifiable curve is not unique in general. If the Hausdorff dimension $\mathrm{Dmh}\,\Gamma$ of a curve Γ exceeds 1, then there exist non-trivial holomorphic in $\overline{\mathbf{C}} \setminus \Gamma$ functions with null jump on Γ (see, for instance, [18]). But if $\mu > \mathrm{Dmh}\,\Gamma - 1$, then any function $\Phi(z)$ satisfying the Hölder condition with exponent μ in a domain $D \supset \Gamma$ and holomorphic in $D \setminus \Gamma$ is holomorphic in D (the E.P. Dolzhenko theorem; see [18]). We say that a holomorphic in $\mathbf{C} \setminus \Gamma$ function $\Phi(z)$ satisfies the Hausdorff–Dolzhenko condition (HD-condition) if curve Γ has a neighborhood N such that restrictions of $\Phi(z)$ on $N \cap D^+$ and $N \cap D^-$ satisfy the Hölder condition with exponent $\mu > \mathrm{Dmh}\,\Gamma - 1$. If a solution of the jump problem (or the Riemann boundary value problem) satisfies HD-condition, then we call it HD-solution. If HD-solution of the jump problem exists, then it is unique. According to Theorem 2.2 the Cauchy transforms give HD-solution of the jump problem if $f \in H_\nu(\Gamma)$ and

$$\mathrm{Dmh}\,\Gamma - 1 < \frac{2\nu - \mathrm{Dma}\,\Gamma}{2 - \mathrm{Dma}\,\Gamma}. \tag{2.2}$$

Let us denote

$$\mathrm{Dmu}\,\Gamma := \mathrm{Dma}\,\Gamma + (2 - \mathrm{Dma}\,\Gamma)(\mathrm{Dmh}\,\Gamma - 1).$$

The meanings of this value are contained between 1 and 2, and we consider it as metric characteristic of dimensional type. The inequality (2.2) is equivalent to $\nu > \mathrm{Dmu}\,\Gamma/2$. Thus, there is valid

Corollary 2.5. *If* $f \in H^*(\Gamma, \mathrm{Dmu}\,\Gamma/2)$*, then the Cauchy transforms* $C[\chi^+]f^w(z)$ (*for* $\mathrm{Dma}\,\Gamma = \mathrm{Dma}^+\,\Gamma$) *and* $C[-\chi^+]f^w(z)$ (*for* $\mathrm{Dma}\,\Gamma = \mathrm{Dma}^-\,\Gamma$) *represent a unique HD-solution of the jump problem* (0.2)*.*

Now we apply the factorization procedure (see [1, 2]) for solving of the Riemann boundary problem (0.1). Let $G(t)$ do not vanish on Γ. We represent G as $G(t) = (t - z_0)^\kappa \exp f(t)$, where $z_0 \in D^+$ and κ is divided by 2π decrement of argument of G on Γ, and solve the jump problem

$$\Psi^+(t) - \Psi^-(t) = f(t), t \in \Gamma.$$

If $G \in H^*(\Gamma, \mathrm{Dma}\,\Gamma)/2)$, then f belongs to the same space, and this problem has a solution $\Psi(z) = C[\pm\chi^\pm]f^w(z)$. Then we put

$$X(z) := \exp \Psi(z), z \in D^+, X(z) := (z - z_0)^{-\kappa} \exp \Psi(z), z \in D^-,$$

and substitute $G(t) = X^+(t)/X^-(t)$ into the relation (0.1). It turns into the jump problem

$$\frac{\Phi^+(t)}{X^+(t)} - \frac{\Phi^-(t)}{X^-(t)} = \frac{g(t)}{X^+(t)}, t \in \Gamma.$$

If $g \in H^*(\Gamma, \mathrm{Dma}\,\Gamma)/2)$, then it has a solution $\Xi(z)$ equaling either $C[\frac{\chi^+}{X}]g^w(z)$ or $C[-\frac{\chi^-}{X}]g^w(z)$. Then function $\Phi_0 := \Xi X$ satisfies the equality (0.1). If $\kappa \geq 0$, then $\Phi_0(\infty) = 0$, i.e., Φ_0 is a solution of the Riemann boundary value problem. If $\kappa > 0$, then sum $\Phi_0(z) + X(z)P(z)$ is solution of the problem for any polynomial $P(z)$ of degree lesser than κ. If $\kappa < 0$, then Φ_0 is a solution under restrictions $\int[\frac{\chi^\pm}{X}]g^w(z)z^{-j-1}dz = 0, j = 1, 2, \ldots, -\kappa$. In addition, if G and g belong to space $H^*(\Gamma, \mathrm{Dmu}\,\Gamma)/2)$, then all these solutions satisfy the HD-condition, and the problem has not other HD-solutions. As a result, we obtain

Theorem 2.6. *If* G *and* g *belong to* $H^*(\Gamma, \frac{1}{2}\,\mathrm{Dma}\,\Gamma)$ *and* $G(t)$ *does not vanish on* Γ*, then the function* $\Phi_0(z) = \Xi(z)X(z)$ *is a solution of the Riemann boundary value problem* (0.1) *for* $\kappa = 0$*. If* $\kappa > 0$*, then the problem has a family of solutions* $\Phi(z) = \Phi_0(z) + X(z)P(z)$*, where* $P(z)$ *is arbitrary algebraic polynomial of degree less than* κ*. If* $\kappa < 0$*, then* $\Phi_0(z)$ *is a solution under* $-\kappa$ *solvability conditions.*

If G *and* g *belong to* $H^*(\Gamma, \frac{1}{2}\,\mathrm{Dmu}\,\Gamma)$*, then all these solutions satisfy the HD-condition, and the problem has not other HD-solutions.*

In order words, if G and g belong to $H^*(\Gamma, \frac{1}{2}\,\mathrm{Dmu}\,\Gamma)$ and $G(t)$ does not vanish on Γ, then the HD-solvability pattern of the problem (0.1) repeats its solvability pattern in classical case of piecewise-smooth curve, and all its HD-solutions and conditions of HD-solvability are representable in terms of the integrations and their Cauchy transforms.

3. Non-rectifiable arcs

Let Γ be non-rectifiable Jordan arc with beginning at the point a_1 and end at the point a_2. We seek a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ such that

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), t \in \Gamma', \tag{3.1}$$

where $\Gamma' := \Gamma \setminus \{a_1, a_2\}$, $\Phi(\infty) = 0$ and $\Phi(z) = O(|z - a_{1,2}|^{-\gamma})$ near $a_{1,2}$, $0 \le \gamma < 1$, $\gamma = \gamma(\Phi)$ (see [1, 2]).

At least two essential groups of facts differ this situation from the case of closed curve. The first group concerns definition of integrations. We have not intrinsic meaning of the integral $\int_\Gamma \xi(t)dt$ over arc Γ for $\xi \in C^\infty(\mathbb{C})$, whereas for closed arcs that meaning is given by the Stokes formula. In particular, the integral over arc does not vanish for $\xi =$ const. The second group is connected with polygonal approximation of the arc. If arc Γ curls in spirals at end points, then we cannot approximate it by polygonal lines with same end points without intersections with Γ at its inner points. We restrict our class of arcs in order to exclude the effect of these circumstances.

We say that a sequence of polygonal lines $G = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots\}$ is polygonal approximation of arc Γ from the left (from the right) if

1. Γ_n begins at the point a_1, ends at the point a_2 and has not another common points with arc Γ, $n = 1, 2, \ldots$;
2. the union $\Gamma \cup \Gamma_n$ bounds finite domain P_n such that $P_{n+1} \subset P_n$ for any n;
3. the direction of Γ is positive direction on ∂P_n for approximation from the left and negative one for approximation from the right, $n = 1, 2, \ldots$;
4. if s_n is area of polygon P_n, then $\lim_{n\to\infty} \operatorname{dist}\{\Gamma, \Gamma_n\} = 0$ and $\lim_{n\to\infty} s_n = 0$.

If arc Γ has polygonal approximations from the left and from the right, then we call it PA-arc. Particularly, PA-arcs have not spiral curls at their ends. As above, we denote by λ_n perimeter of the polygon $\Delta_n := P_{n+1} \setminus \overline{P_n}$, by ω_n the diameter of the most disk contained in polygon Δ_n, and call the sum

$$M_d(G) = \sum_{n=1}^{\infty} \lambda_n \omega_n^{d-1}$$

d-mass of the approximation G.

Definition 3.1. Let $A^+(\Gamma)$ (correspondingly, $A^-(\Gamma)$) be set of all positive numbers d such that PA-arc Γ has polygonal approximation from the left (correspondingly, from the right) with finite d-mass. Then the values $\operatorname{Dma}^+\Gamma := \inf A^+(\Gamma)$ and $\operatorname{Dma}^-\Gamma := \inf A^-(\Gamma)$ are left and right approximation dimensions of this arc.

The following result can be proved in just the same way as Theorem 1.2.

Theorem 3.2. *Any PA-arc Γ satisfies inequalities*

$$1 \le \operatorname{Dma}^\pm \Gamma \le \operatorname{Dm}\Gamma \le 2. \tag{3.2}$$

For any value $d \in (1, 2)$ there exist PA-arcs $\Gamma_{1,2}$ such that $\operatorname{Dm}\Gamma = d$, $\operatorname{Dma}^+\Gamma_1 < d$ and $\operatorname{Dma}^-\Gamma_2 < d$.

Let Γ_n belongs to a polygonal approximation G of non-rectifiable PA-arc Γ from the left. Then $\Gamma \cup \Gamma_n$ is boundary of finite domain P_n. The direction of Γ is positive direction on ∂P_n, and direction of Γ_n from a_1 to a_2 is negative. If Γ would be rectifiable, then

$$\int_\Gamma \varphi(\zeta)d\zeta = \int_{\Gamma\cup\Gamma_n} \varphi(\zeta)d\zeta + \int_{\Gamma_n} \varphi(\zeta)d\zeta.$$

If Γ is non-rectifiable, then we can replace the first term in the right side by the primary integration from the Section 1. Thus, we put

$$\int_\Gamma [F]\varphi d\zeta := \langle \overline{\partial} F, \varphi\rangle + \int_{\Gamma_n} (F^+(\zeta) - F^-(\zeta))\varphi(\zeta)d\zeta,$$

where $\varphi \in C_0^\infty$, $F(\zeta)$ is holomorphic in $\overline{\mathbb{C}}\setminus(\Gamma\cup\Gamma_n)$ and $F(\infty) = 0$. This value does not depend on n. The polygonal approximation from the right induces analogous construction for primary integration over non-rectifiable arc.

Let us cite three representations of primary integration over PA-arc Γ with unit weight.

1. Let $F(\zeta) = \chi+(\zeta)$ be characteristic function of domain P_n. Then
$$\int_\Gamma [\chi^+]\varphi(\zeta)d\zeta := -\iint_{P_n} \frac{\partial\varphi}{\partial\overline{\zeta}} d\zeta d\overline{\zeta} + \int_{\Gamma_n} \varphi(\zeta)d\zeta.$$
We can take φ from the whole space $C^\infty(\mathbb{C})$.
2. Analogously, if $F(\zeta) = -\chi-(\zeta)$, where $\chi-(\zeta)$ is characteristic function of domain P_n induced by a polygonal approximation from the right, then
$$\int_\Gamma [-\chi^-]\varphi(\zeta)d\zeta := -\iint_{P_n} \frac{\partial\varphi}{\partial\overline{\zeta}} d\zeta d\overline{\zeta} + \int_{\Gamma_n} \varphi(\zeta)d\zeta.$$
3. Let
$$k(\zeta) = \frac{1}{2\pi i}\log\frac{\zeta - a_2}{\zeta - a_1},$$
where the branch of logarithm is selected by means of cut along the arc Γ and restriction $k(\infty) = 0$. The jump of the kernel k on Γ equals to 1, and on Γ_n its jump vanishes. Thus,
$$\int_\Gamma [k]\varphi(\zeta)d\zeta := -\iint_{P_n} k(\zeta)\frac{\partial\varphi}{\partial\overline{\zeta}} d\zeta d\overline{\zeta},$$
where $\varphi \in C_0^\infty(\mathbb{C})$. This is also generalization of integral over Γ with unit weight.

Let us note in connection with the last representation that for PA-arc Γ the kernel $k(\zeta)$ has logarithmic singularities at the points $a_{1,2}$. For general non-rectifiable arcs these singularities can have arbitrarily high order.

Now we can repeat considerations of previous two sections for PA-arcs. Particularly, we obtain

Theorem 3.3. *Let* Γ *be* PA*-arc,* $G(t) \equiv 1$*, and* $g \in H^*(\Gamma, \operatorname{Dma}\Gamma/2)$*. Then the problem* (3.1) (*i.e., the jump problem*) *has a solution representable by the Cauchy transform of corresponding integration.*

As above, $\operatorname{Dma}\Gamma$ stands for $\min\{\operatorname{Dma}^+\Gamma, \operatorname{Dma}^-\Gamma\}$. Earlier (see [19]) the solvability of jump problem was established for $g \in H^*(\Gamma, \operatorname{Dm}\Gamma/2)$ in terms of the present paper.

One can easily reformulate the HD-condition for arcs. As above, if G and g belong to $H^*(\Gamma, \frac{1}{2}\operatorname{Dmu}\Gamma)$ and $G(t)$ does not vanish on PA-arc Γ, then the HD-solvability pattern of the problem (0.1) repeats its solvability pattern in classical case of piecewise-smooth arc, and all its HD-solutions and conditions of HD-solvability are representable in terms of the integrations and their Cauchy transforms.

We consider also the following question. Let $p(z) = \sum_{j=0}^{m} p_j z^j$ be algebraic polynomial. If an arc Γ is rectifiable, then $|p(a_2) - p(a_1)| \le C \max\{|p'(z)| : z \in \Gamma\}$, where positive constant C does not depend on p, and the best meaning of this constant is length of Γ. It is of interest to find a functional Banach space X such that $|p(a_2) - p(a_1)| \le C\|p'(z)\|_X$ for non-rectifiable arc Γ. Here we use to this end the Hölder spaces $H_\nu(\Gamma)$.

Theorem 3.4. *Let* Γ *be* PA*-arc and* $\nu > \operatorname{Dma}\Gamma - 1$*. Then any algebraic polynomial* $p(z)$ *satisfies inequality*

$$|p(a_2) - p(a_1)| \le C\|p'\|_{H_\nu}, \tag{3.3}$$

where $\|p'\|_{H_\nu} = h_\nu(p', \Gamma) + \max\{|p'(z)| : z \in \Gamma\}$*, and the constant* C *depends on* Γ *and* ν *only.*

Proof. Assume that $\operatorname{Dma}\Gamma = \operatorname{Dma}^+\Gamma$. Then we can fix a value d such that $\nu > d - 1$ and Γ has polygonal approximation from the left $G = \{\Gamma_1, \Gamma_2, \dots\}$ with finite d-mass. We denote $\Gamma^* = \overline{\cup_{n=1}^{\infty}\Gamma_n}$. Let p^w be the Whitney extension of p' from Γ on the whole $\mathbb{C}$, and p^* the Whitney extension of restriction $p^w|_{\Gamma^*}$. By virtue of definitions of the Whitney extension and the polygonal approximation we have $\lim_{n\to\infty}\int_{\Gamma_n}(p'(\zeta) - p^w(\zeta))d\zeta = 0$. We obtain $\lim_{n\to\infty}\int_{\Gamma_n} p^w(\zeta)d\zeta = p(a_2) - p(a_1)$. Hence,

$$p(a_2) - p(a_1) = -\iint_{P_n} \frac{\partial p^*}{\partial\overline{\zeta}} d\zeta d\overline{\zeta} + \int_{\Gamma_n} p^*(\zeta)d\zeta \tag{3.4}$$

for any n. By virtue of well-known properties of the Whitney extension we have $\max\{|p^*(\zeta)| : \zeta \in \Gamma^*\} = \max\{|p'(\zeta)| : \zeta \in \Gamma\}$, and $|\int_{\Gamma_n} p^*(\zeta)d\zeta| \le L_n \max\{|p'(\zeta)| : \zeta \in \Gamma\} \le L_n\|p'\|_{H_\nu}$, where L_n is length of Γ_n. Then we bound the first term in right side of (3.4) by means of Lemma 1.5. As in the proof of Theorem 1.4, we obtain

$$\left|\iint_{P_n} \frac{\partial p^*}{\partial\overline{\zeta}} d\zeta d\overline{\zeta}\right| \le c(\nu, d) s_n^{1/q} M_d^{1/p}(G_n) h_\nu(p', \Gamma),$$

where $c(\nu, d)$ depends on ν and d only, G_n is subsequence $\{\Gamma_n, \Gamma_{n+1}, \dots\}$, s_n is area of P_n, the value p is defined by equality (1.2), and $p^{-1} + q^{-1} = 1$. Consequently,

$$|p(a_2) - p(a_1)| \le c(\nu, d)(L_n + s_n^{1/q} M_d^{1/p}(G_n))\|p'\|_{H_\nu}$$

for any n. □

We say that arc Γ is *H_ν-rectifiable* if the functional $p' \mapsto p(a_2) - p(a_1)$ is bounded in $H_\nu(\Gamma)$. We proved that PA-arc is H_ν-rectifiable for $\nu > \operatorname{Dma}\Gamma - 1$. In this connection we call the best value of constant C in (3.3) *H_ν-length* of this arc. We see that H_ν-length of Γ does not exceed the value

$$c(\nu, d) \inf_G \inf_n \{L_n + s_n^{1/q} M_d^{1/p}(G_n)\},$$

where the most lower bound is taken first over $n = 1, 2, \dots$, and then over all approximations G with finite d-mass.

Earlier the H_ν-rectifiability of arcs without spiral curls at end-points was proved for $\nu > \operatorname{Dm}\Gamma - 1$ in the paper [20].

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Bloch Solutions of Periodic Dirac Equations in SPPS Form

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Abstract. We provide the representation of quasi-periodic solutions of periodic Dirac equations in terms of the spectral parameter power series (SPPS) recently introduced by V.V. Kravchenko [1, 2, 3]. We also give the SPPS form of the Dirac Hill discriminant under the Darboux nodeless transformation using the SPPS form of the discriminant. and apply the results to one of Razavy's quasi-exactly solvable periodic potentials.

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1. Introduction

The connections between the Dirac equation and the Schrödinger equation are known since a long time ago [4] and have been strengthen in the supersymmetric context soon after the advent of supersymmetric quantum mechanics in 1981 [5, 6, 7, 8]. There are currently interesting applications of this approach in condensed matter physics [9, 10, 11]. In this work, we are interested in the same connection in the case of periodic potentials, see, e.g., [12]. We here write the Dirac Bloch solutions in Kravchenko form (power series in the spectral parameter) and also the Dirac Hill discriminant in the same form and apply the results to an interesting quasi-exactly solvable periodic potential.

2. Schrödinger equations of Hill type

The Schrödinger differential equation

$$L\left[f(x,\lambda)\right] = -f''(x,\lambda) + q(x)f(x,\lambda) = \lambda f(x,\lambda) \qquad (2.1)$$

with T-periodic real-valued potential $q(x)$ assumed herewith a continuous bounded function and λ a real parameter is known as of Hill type. We begin by recalling some necessary definitions and basic properties associated with the equation (2.1) from the Floquet (Bloch) theory. For more details see, e.g., [13, 14].

For each λ there exists a fundamental system of solutions, i.e., two linearly independent solutions of (2.1), $f_1(x,\lambda)$ and $f_2(x,\lambda)$, which satisfy the initial conditions

$$f_1(0,\lambda) = 1, \quad f_1'(0,\lambda) = 0, \quad f_2(0,\lambda) = 0, \quad f_2'(0,\lambda) = 1. \tag{2.2}$$

Then the Hill discriminant associated with equation (2.1) is defined as a function of λ as follows

$$D(\lambda) = f_1(T,\lambda) + f_2'(T,\lambda).$$

The importance of $D(\lambda)$ stems from the easiness of describing the spectrum of the corresponding equation by its means, namely [13]:

(1) sets $\{\lambda_i\}$ for which $|D(\lambda)| \leq 2$ form the allowed bands or stability intervals,
(2) sets $\{\lambda_j\}$ for which $|D(\lambda)| > 2$ form the forbidden bands or instability intervals,
(3) sets $\{\lambda_k\}$ for which $|D(\lambda)| = 2$ form the band edges and represent the discrete part of the spectrum.

Furthermore, when $D(\lambda) = 2$ equation (2.1) has a periodic solution with the period T and when $D(\lambda) = -2$ it has an aperiodic solution, i.e., $f(x+T) = -f(x)$. The eigenvalues λ_n, $n = 0, 1, 2, \ldots$ form an infinite sequence $\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 \cdots$, and an important property of the minimal eigenvalue λ_0 is the existence of a corresponding periodic nodeless solution $u(x,\lambda_0)$ [13]. The solutions of (2.1) are not periodic in general, and one of the important tasks is the construction of quasiperiodic solutions defined by $f_\pm(x+T) = \beta_\pm(\lambda) f_\pm(x)$. Here, we use James' matching procedure [15] that employs the fundamental system of solutions, $f_1(x,\lambda)$ and $f_2(x,\lambda)$, in the construction of the quasiperiodic solutions as follows

$$f_\pm(x,\lambda) = \beta_\pm^n(\lambda)\left[f_1(x-nT,\lambda) + \alpha_\pm f_2(x-nT,\lambda)\right], \quad \left\{ \begin{array}{l} nT \leq x < (n+1)T \\ n = 0, \pm 1, \pm 2, \ldots \end{array} \right., \tag{2.3}$$

where $\alpha_\pm$ are given by [15]

$$\alpha_\pm = \frac{f_2'(T,\lambda) - f_1(T,\lambda) \mp \left(D^2(\lambda) - 4\right)^{\frac{1}{2}}}{2 f_2(T,\lambda)}\,. \tag{2.4}$$

The Bloch factors $\beta_\pm(\lambda)$ are a measure of the rate of increase (or decrease) in magnitude of the linear combination of the fundamental system when one goes from the left end of the cell to the right end, i.e.,

$$\beta_\pm(\lambda) = \frac{f_1(T,\lambda) + \alpha_\pm f_2(T,\lambda)}{f_1(0,\lambda) + \alpha_\pm f_2(0,\lambda)}\,.$$

The values of $\beta_\pm(\lambda)$ are directly related to the Hill discriminant, $\beta_\pm(\lambda) = \frac{1}{2}(D(\lambda) \mp \sqrt{D^2(\lambda) - 4})$, and obviously at the band edges $\beta_+ = \beta_- = \pm 1$ for $D(\lambda) = \pm 2$, respectively.

3. SPPS representation for solutions of the one-dimensional Dirac equation

We consider the following Dirac equation

$$L[W] = [-i\sigma_y d_x + \sigma_x \Phi(x)]\, W = \omega W, \tag{3.1}$$

where the scalar potential $\Phi(x)$ is periodic function with period T, W is the spinor $W = \begin{pmatrix} f \\ g \end{pmatrix}$ and σ_x, σ_y are the Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

The uncoupled Schrödinger equations derived from equation (3.1) are

$$(-d_x + \Phi)(d_x + \Phi)f = \lambda f, \tag{3.2}$$

$$(d_x + \Phi)(-d_x + \Phi)g = \lambda g, \tag{3.3}$$

where $\lambda = \omega^2$ is the spectral parameter. It is clear that the solutions f and g are related by the following relationship

$$(d_x + \Phi)f = \omega g, \tag{3.4}$$

therefore with the solution f at hand, we can construct the solution g immediately.

We start with equation (3.2). Notice that the solution u of the equation (3.2) for $\lambda = 0$ can be obtained as follows $u(x) = e^{-\int \Phi(x)dx}$ and $u(x)$ is a nodeless periodic function with the period T if $\Phi(x) \in \mathbf{C}^1$ and $\int_0^T \Phi(x)dx = 0$.

Once having the function $u(x)$ the solutions $f_1(x,\lambda)$ and $f_2(x,\lambda)$ of (3.2), (2.2) for all values of the parameter λ can be given using the SPPS method [1].

$$\begin{aligned} f_1(x,\lambda) &= \frac{u(x)}{u(0)}\widetilde{\Sigma}_0(x,\lambda) + u'(0)u(x)\Sigma_1(x,\lambda), \\ f_2(x,\lambda) &= -u(0)u(x)\Sigma_1(x,\lambda). \end{aligned} \tag{3.5}$$

The functions $\widetilde{\Sigma}_0$ and Σ_1 are the spectral parameter power series

$$\widetilde{\Sigma}_0(x,\lambda) = \sum_{n=0}^{\infty} \widetilde{X}^{(2n)}(x)\lambda^n, \quad \Sigma_1(x,\lambda) = \sum_{n=1}^{\infty} X^{(2n-1)}(x)\lambda^{n-1}\ ,$$

where the coefficients $\widetilde{X}^{(n)}(x)$, $X^{(n)}(x)$ are given by the following recursive relations

$$\widetilde{X}^{(0)} \equiv 1, \qquad X^{(0)} \equiv 1,$$

$$\tilde{X}^{(n)}(x) = \begin{cases} \displaystyle\int_0^x \tilde{X}^{(n-1)}(\xi)u^2(\xi)d\xi & \text{for an odd } n \\ \displaystyle -\int_0^x \tilde{X}^{(n-1)}(\xi)\frac{d\xi}{u^2(\xi)} & \text{for an even } n \end{cases} \tag{3.6}$$

$$X^{(n)}(x) = \begin{cases} -\int_0^x X^{(n-1)}(\xi)\frac{d\xi}{u^2(\xi)} & \text{for an odd } n \\ \int_0^x X^{(n-1)}(\xi)u^2(\xi)d\xi & \text{for an even } n \ . \end{cases} \tag{3.7}$$

One can check by a straightforward calculation that the solutions f_1 and f_2 fulfill the initial conditions (2.2), for this the following relations are useful

$$\left(\widetilde{\Sigma}_0(x,\lambda)\right)'_x = -\frac{\widetilde{\Sigma}_1(x,\lambda)}{u^2(x)}, \quad \text{where} \quad \widetilde{\Sigma}_1(x,\lambda) = \sum_{n=1}^{\infty} \widetilde{X}^{(2n-1)}(x)\lambda^n \tag{3.8}$$

and

$$\left(\Sigma_1(x,\lambda)\right)'_x = -\frac{\Sigma_0(x,\lambda)}{u^2(x)}, \quad \text{where} \quad \Sigma_0(x,\lambda) = \sum_{n=0}^{\infty} X^{(2n)}(x)\lambda^n. \tag{3.9}$$

The pair of linearly independent solutions $g_1(x,\lambda)$ and $g_2(x,\lambda)$ of (3.3) can be obtained directly from the solutions (3.5) by means of (3.4). We additionally take the linear combinations in order that the solutions $g_1(x,\lambda)$ and $g_2(x,\lambda)$ satisfy the initial conditions $g_1(0,\lambda) = g_2'(0,\lambda) = 1$ and $g_1'(0,\lambda) = g_2(0,\lambda) = 0$

$$\begin{aligned} g_1(x,\lambda) &= \frac{u(0)}{u(x)}\Sigma_0(x,\lambda) - \frac{\Phi(0)}{\lambda u(0)u(x)}\widetilde{\Sigma}_1(x,\lambda), \\ g_2(x,\lambda) &= \frac{1}{\lambda u(0)u(x)}\widetilde{\Sigma}_1(x,\lambda). \end{aligned} \tag{3.10}$$

Thus, the two spinor solutions of the Dirac equation (3.1) are given by

$$W_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \quad \text{and} \quad W_2 = \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$$

and these solutions satisfy the following initial conditions

$$W_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad W_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

3.1. Bloch solutions and Hill discriminant

The second-order differential equations (3.2) and (3.3) have periodic potentials $V_{1,2} = \Phi^2 \mp \Phi'$, correspondingly. The important tasks for this case are the construction of the Bloch solutions which are subject to the Bloch condition $f(x+T) = e^{iTx}f(x)$ (with a wave number k) and the description of the spectrum.

In [16] the SPPS representations of Hill discriminants $D_f(\lambda)$ and $D_g(\lambda)$ associated with the equations (3.2) and (3.3) were obtained in the form

$$\begin{aligned} D_f(\lambda) &= \frac{u(T)}{u(0)}\widetilde{\Sigma}_0(T,\lambda) + \frac{u(0)}{u(T)}\Sigma_0(T,\lambda) + \left(u'(0)u(T) - u(0)u'(T)\right)\Sigma_1(T,\lambda)\ , \\ D_g(\lambda) &= \frac{u(0)}{u(T)}\Sigma_0(T,\lambda) + \frac{u(T)}{u(0)}\widetilde{\Sigma}_0(T,\lambda) + \frac{1}{(\Delta\lambda)\, u^2(0)u^2(T)}\left(u'(0)u(T)\right. \\ &\qquad \left. -u'(T)u(0)\right)\widetilde{\Sigma}_1(T,\lambda). \end{aligned}$$

It is clear that since $u(x)$ is a T-periodic function ($u(0) = u(T)$) the expression in brackets in the above formulae vanishes. Now writing the explicit expressions for $\widetilde{\Sigma}_0(T, \lambda, \lambda_0)$ and $\Sigma_0(T, \lambda, \lambda_0)$, a representation for Hill's discriminant associated with (3.2) and (3.3) is the following

$$D_f(\lambda) = D_g(\lambda) = \sum_{n=0}^{\infty} \left(\tilde{X}^{(2n)}(T) + X^{(2n)}(T) \right) \lambda^n. \tag{3.11}$$

Equations (3.2) and (3.3) are isospectral and we obtain the Hill discriminant associated with the Dirac equation (3.1). We formulate this result as the following theorem:

Theorem 3.1. *Let $\Phi(x) \in \mathbf{C}^1$ be a T-periodic function which satisfies the condition $\int_0^T \Phi(x)dx = 0$. Then the Hill discriminant for (3.1) has the form*

$$D_W(\omega) = \sum_{n=0}^{\infty} \left(\tilde{X}^{(2n)}(T) + X^{(2n)}(T) \right) \omega^{2n} ,$$

where $\tilde{X}^{(2n)}$ and $X^{(2n)}$ are calculated according to (3.6) and (3.7), $u = e^{-\int \Phi(x)dx}$ and the series converges uniformly on any compact set of values of ω.

In order to construct the Bloch solutions for the Dirac equation (3.1) we use the solutions (3.5) and (3.10) and apply the procedure of James [15]. Notice that because the Hill discriminants for the equations (3.2) and (3.3) are identical the Bloch factors for both equations are equal. The so-called self-matching solutions for the equations (3.2) and (3.3), are correspondingly

$$F_\pm(x, \lambda) = f_1(x, \lambda) + a_\pm f_2(x, \lambda) \text{ and } G_\pm(x, \lambda) = g_1(x, \lambda) + b_\pm g_2(x, \lambda),$$

where $a_\pm$ and $b_\pm$ are calculated by the formula (2.4) with the corresponding fundamental system of solutions (3.5) and (3.10). By means of $F_\pm$ and $G_\pm$ we write the self-matching spinor solution of the equation (3.1)

$$w_\pm(x, \lambda) = \begin{pmatrix} F_\pm(x, \lambda) \\ G_\pm(x, \lambda) \end{pmatrix}.$$

Finally, the Bloch solutions of the equation (3.1) take the form

$$W_\pm(x, \lambda) = \beta_\pm^n(\lambda) \left(w_\pm(x - nT, \lambda) \right), \quad \begin{cases} nT \le x < (n+1)T \\ n = 0, \pm 1, \pm 2, \dots \end{cases} .$$

4. Numerical calculation of eigenvalues based on the SPPS form of Hill's discriminant

As is well known [13], the zeros of the functions $D(\lambda) \mp 2$ represent eigenvalues of the corresponding Hill operator with periodic and aperiodic boundary conditions, respectively. In this section, we show that besides other possible applications the representation (3.11) gives us an efficient tool for the calculation of the discrete spectrum of a periodic Dirac operator.

The first step of the numerical realization of the method consists in calculation of the functions $\tilde{X}^{(n)}$ and $X^{(n)}$ given by (3.6) and (3.7), respectively. This construction is based on the eigenfunction $u(x)$. Next, by truncating the infinite series for $D(\lambda)$ (3.11) we obtain a polynomial in λ

$$D_N(\lambda) = \sum_{n=0}^{N} \left(\tilde{X}^{(2n)}(T) + X^{(2n)}(T) \right) \lambda^n = 2 + \sum_{n=1}^{N} \left(\tilde{X}^{(2n)}(T) + X^{(2n)}(T) \right) \lambda^n. \tag{4.1}$$

The roots of the polynomials $D_N(\lambda) \mp 2$ give us the eigenvalues corresponding to equation (2.1) with periodic and aperiodic boundary conditions, respectively.

As an example we consider the Dirac equation (3.1) with the scalar potential

$$\Phi(x) = \sin 2x \left[\frac{\xi}{2} - \frac{2A(\xi)}{\xi - A(\xi)\cos 2x} \right] ,$$

with $A(\xi) = \left(1 - \sqrt{1+\xi^2}\right)$ and ξ a real positive parameter. This scalar potential satisfies the conditions of theorem 3.1. The corresponding second-order differential equations are

$$-d_x^2 f + V_1 f = \lambda f,$$
$$-d_x^2 g + V_2 g = \lambda g,$$

where the Schrödinger potential

$$V_1(x) = \frac{\xi^2}{8}(1 - \cos 4x) - 3\xi \cos 2x \ , \tag{4.2}$$

is the case $m = 2$ in the quasi-exactly solvable family of the so-called trigonometric Razavy potentials [17], $V_R = \frac{\xi^2}{8}(1 - \cos 4x) - (m+1)\xi \cos 2x$. For a given integer m, if $\xi < 2(m+1)$ the potentials $V_R(x)$ are of single-well periodic type and if $\xi > 2(m+1)$ they are of double-well periodic type.

$$V_2(x) = V_1(x) + 4\cos 2x \left(\frac{\xi}{2} - \frac{2A(\xi)}{\xi - A(\xi)\cos 2x} \right) + \frac{8A(\xi)\sin^2 2x}{(\xi - A(\xi)\cos 2x)^2} \tag{4.3}$$

is the supersymmetric partner potential and therefore it is also quasi-exactly solvable. The Schrödinger equations with these potentials can be used for the description of torsional oscillations of certain molecules [17]. Plots of the potentials $V_1(x)$ and $V_2(x)$ are displayed in Figure 1 for two values of ξ.

The computer algorithm was implemented in Matlab 2006. The recursive integration required for the construction of $\tilde{X}_0^{(n)}$, $X_0^{(n)}$, $\tilde{X}^{(n)}$ and $X^{(n)}$ was done by representing the integrand through a cubic spline using the *spapi* routine with a division of the interval $[0, \pi]$ into 5000 subintervals and integrating using the *fnint* routine. Next, the zeros of $D_N(\lambda) \pm 2$ were calculated by means of the *fnzeros* routine.

In the following tables, the eigenvalues were calculated employing the SPPS representation (3.11) for four different values of the parameter ξ. The first two

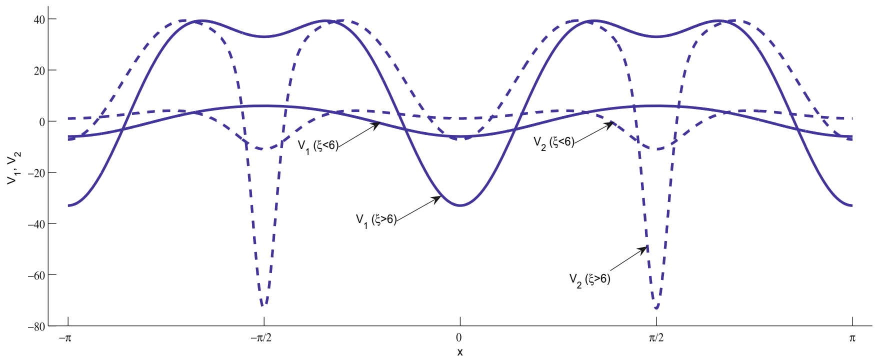

FIGURE 1. The Razavy potentials V_1 (solid lines) for $\xi = 2$ and $\xi = 11$ given by (4.2) and its partner potentials V_2 (dashed lines) as given by (4.3) for the same values of ξ.

values are below the threshold value $\xi_{\text{thr}} = 6$ for $m = 2$ from single-well to double-well types of Razavy's potentials while the last two values are above this threshold value. For comparison, we use the eigenvalues given analytically by Razavy in terms of the parameter ξ as follows [17]

$$\lambda_0 = 2\left(1 - \sqrt{1+\xi^2}\right), \quad \lambda_3 = 4, \quad \lambda_4 = 2\left(1 + \sqrt{1+\xi^2}\right) .$$

	$\xi = 1$	$\xi = 1$
n	λ_n (SPPS)	λ_n (Ref. [17])
0	−0.828427124746190	−0.828427124746190
1	−0.628906956748252	
2	2.315132548422588	
3	3.999991462865745	4
4	4.828420096225068	4.828427124746190
5	9.238264469324272	
6	9.294265517212145	

	$\xi = 2$	$\xi = 2$
n	λ_n (SPPS)	λ_n (Ref. [17])
0	−2.472135954999580	−2.472135954999580
1	−2.428136886851045	
2	3.184130151531468	
3	4.000004180961838	4
4	6.472138385406806	6.472135954999580
5	9.864117523158974	
6	10.253256926576858	

	$\xi = 11$	$\xi = 11$
n	λ_n (SPPS)	λ_n (Ref. [17])
0	−20.090722034374522	−20.090722034374522
1	−20.090721031408926	
2	3.999728397824670	
3	4.000000543012631	4
4	24.092379855485746	24.090722034374522
5	24.125593160436161	
6	36.212102534969766	

	$\xi = 20$	$\xi = 20$
n	λ_n (SPPS)	λ_n (Ref. [17])
0	−38.049968789001575	−38.049968789001575
1	−38.049968788934475	
2	3.999999942823312	
3	3.999999999630503	4
4	42.050313148383374	42.049968789001575
5	42.050347742353317	
6	74.691604620863302	

In Figure 2, we display the plots of the Hill discriminants for the values of the Razavy parameter $\xi = 1$, $\xi = 2$, and $\xi = 3$, respectively. In general, these plots contain damped oscillations with higher amplitudes at higher ξ. On the other hand, getting the spectrum in λ is equivalent with having the eigenvalues $\omega_n = \pm\sqrt{\lambda_n}$ of the Dirac system under consideration.

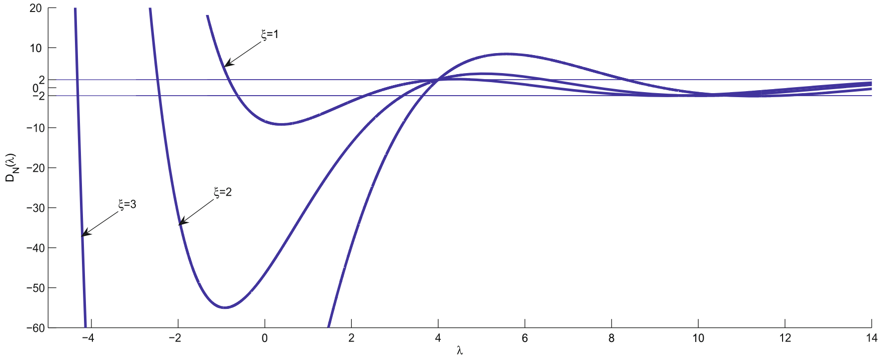

FIGURE 2. The polynomial $D_N(\lambda)$ for the Hill equations with Razavy's partner potentials for three values of the parameter ξ calculated by means of formula (4.1) for $N = 100$.

5. Conclusions

In summary, in this work we presented the SPPS form of the quasi-periodic (Bloch) solutions of periodic one-dimensional Dirac operators as well as of the Hill discriminant. We applied the obtained results to the Dirac system with the periodic scalar potential that leads to one of Razavy's quasi-exactly solvable periodic potentials.

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An Estimate for the Number of Solutions of a homogeneous Generalized Riemann Boundary Value Problem with Shift

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Abstract. The generalized Riemann boundary value problem with the condition on the real line $\varphi_+ = a\varphi_- + a_0\overline{\varphi_-} + a_1\overline{\varphi_-(\alpha)} + a_2\overline{\varphi_-(\alpha_2)} + \cdots + a_m\overline{\varphi_-(\alpha_m)}$, where $\alpha(t) = t + \mu$, $\mu \in \mathbb{R}$, is the shift on the real line, $\alpha_k(t) = t + k\mu$, $0 \le k \le m$, $k, m \in \mathbb{N}$, is considered. Under certain conditions on the coefficients a, a_k, $0 \le k \le m$, an estimate for the number of linearly independent solutions of this problem is obtained.

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Keywords. Singular integral operators, shift operators.

1. Introduction

On $\mathring{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, the one-point compactification of the real line, we consider the non-Carleman shift,

$$\alpha(t) = t + \mu, \quad t \in \mathring{\mathbb{R}},$$

where μ is a fixed real number. The k-iteration of the shift is $\alpha_k(t) = t + k\mu$, $0 \le k \le m$, $k, m \in \mathbb{N}$, and we assume that $\alpha_0(t) \equiv t$. The shift α has the only fixed point at infinity.

Let $a, a_0, a_1, \ldots, a_m \in C(\mathring{\mathbb{R}})$, be given continuous functions defined on $\mathring{\mathbb{R}}$. As usual, let $\widetilde{L}_2(\mathbb{R})$ denote the real space of all Lebesgue measurable square summable complex-valued functions on $\mathbb{R}$. We consider the generalized Riemann boundary value problem: find the functions $\varphi_+(z)$ and $\varphi_-(z)$, analytic in the upper

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and in the lower half-planes, respectively, with boundary values in $\widetilde{L}_2(\mathbb{R})$ satisfying the condition

$$\begin{aligned}\varphi_+(t) &= a(t)\varphi_-(t) + a_0(t)\overline{\varphi_-(t)} + a_1(t)\overline{\varphi_-(\alpha(t))}\\ &\quad + a_2(t)\overline{\varphi_-(\alpha_2(t))} + \cdots + a_m(t)\overline{\varphi_-(\alpha_m(t))}.\end{aligned} \tag{1}$$

Now let us consider in $\widetilde{L}_2(\mathbb{R})$ the following operators: the identity operator I, the isometric non-Carleman shift operator

$$(U\varphi)(t) = \varphi(t+\mu),$$

the linear operator of complex conjugation

$$(C\varphi)(t) = \overline{\varphi(t)},$$

the operator of singular integration with Cauchy kernel

$$(S\varphi)(t) = (\pi i)^{-1}\int_{\mathbb{R}} \varphi(\tau)(\tau - t)^{-1}d\tau,$$

the mutually complementary projection operators

$$P_\pm = \frac{1}{2}(I \pm S),$$

the functional operator

$$A = \sum_{j=0}^{m} a_j U^j,$$

and the singular integral operator with shift and conjugation

$$K_1 = -P_+ + (aI + AC)P_-. \tag{2}$$

Let n denote the number of linearly independent solutions of the problem (1); it is clear that $n = \dim\ker K_1$.

The study of the problem (1) leads to the study of the kernel of the operator (2). Analogously we could study the related singular integral equation, obtained by applying the Fourier Transform to the equation $K_1 f = 0$ (with $a = -1$),

$$\int_{\mathbb{R}^-} \sum_{j=0}^{m} \widehat{a}_j(t-\tau)e^{i\mu\tau}\widehat{\overline{f(\tau)}}d\tau - \widehat{f}(t) = 0,$$

where $\widehat{a}_j$, $\widehat{\overline{f}}$, $\widehat{f}$, are the Fourier transforms of the functions a_j, $\overline{f}$, f, respectively.

The history of boundary value problems with shift, as well as singular integral equations with shift, and related singular integral operators, is rich. These problems were studied during the last fifty years, particularly in the sixties and the seventies of the XXth century, when the theory of this type of boundary value problems was actively pursued. Ilya Vekua's book [21] (first edition in 1959) played a key role in this process; in this and in other similar books (see, e.g., [22]), it has been shown how some mathematical physics problems lead to the solvability of boundary value problems with shift. The Fredholm theory of boundary value problems with Carleman shift, i.e., a diffeomorpfism of a curve onto itself, which after a finite number of iterations, coincide with the identity transform, was constructed in the decades mentioned [16]. For the case of non-Carleman shift, the theory was completed in the eighties [7]. However, more interesting questions about the solvability of boundary value problems with shifts, have been considered only with very restrictive conditions on the respective coefficients [16]. Recent progress in the study of the spectral properties of singular integral operators with linear fractional Carleman shift and conjugation (see [4], [8], [9], [10] and [11]) makes it possible to study the solvability of the related boundary value problems [17]. For non-Carleman shift, the question about the solvability of this type of problems remains open (see [12], [13] and [19]).

In [14] we studied the generalized Riemann boundary value problem (1) with a non-Carleman shift and conjugation, with $a_2 = a_3 = \cdots = a_m = 0$. For the number of linearly independent solutions of this problem, the following estimate

$$n \leq l(F) + \max(k_a - k, 0) + \max(k_a + k, 0)$$

was obtained (see formula (19) below). We had noted that the influence of the coefficient a_1 is restricted to the term $l(F)$; the terms k_a and k depend only on the coefficients a and a_0. In the present paper we consider the problem (1), with iterations of the shift and conjugation. It is interesting to note that the influence of the coefficients $a_1, a_2, \ldots, a_m$ is again restricted to the term $l(F)$ only. The estimate (19) for the number of linearly independent solutions of the problem (1) is obtained. Then we consider a particular case which shows that, in a sense, our estimate is sharp.

2. Main result

Proposition 2.1. *Let $K_2 : \widetilde{L}_2^2(\mathbb{R}) \to \widetilde{L}_2^2(\mathbb{R})$ be the paired operator with shift*

$$K_2 = M_1 P_+ + M_2 P_-, \tag{3}$$

where M_1, M_2, are the functional operators

$$M_1 = \begin{pmatrix} -1 & A \\ 0 & \overline{a} \end{pmatrix}, \quad M_2 = \begin{pmatrix} a & 0 \\ \widetilde{A} & -1 \end{pmatrix},$$

with

$$\widetilde{A} = \sum_{j=0}^{m} \overline{a_j} U^j;$$

then

$$n = \frac{1}{2} \dim \ker K_2.$$

Proof. Making use of the properties

$$C^2 = I, \quad CU = UC, \quad UP_\pm = P_\pm U, \quad CP_\pm = P_\mp C,$$

we obtain the following relation between the operators K_1 and K_2

$$N_1 \mathrm{diag}\,(K_1, \widetilde{K}_1) N_1^{-1} = K_2, \quad \text{where} \quad \widetilde{K}_1 = -P_+ + (aI - AC)P_-,$$

and N_1 is the following invertible operator in $\widetilde{L}_2^2(\mathbb{R})$

$$N_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ C & -C \end{pmatrix}.$$

We then have

$$\dim\ker K_1 + \dim\ker \widetilde{K}_1 = \dim\ker K_2.$$

Since $(iI)^{-1}K_1(iI) = \widetilde{K}_1$, then $\dim\ker K_1 = \dim\ker \widetilde{K}_1$.

Thus

$$n = \dim\ker K_1 = \frac{1}{2}\dim\ker K_2. \qquad \square$$

Assume that $a \in C(\overset{\circ}{\mathbb{R}})$, $a(t) \neq 0$, everywhere on $\overset{\circ}{\mathbb{R}}$. Then the functional operators M_1, M_2 are invertible, so the operator K_2 is Fredholm in $L_2^2(\mathbb{R})$ [7].

Note that

$$M_2^{-1} = \begin{pmatrix} a^{-1} & 0 \\ \widetilde{A}a^{-1} & -1 \end{pmatrix},$$

and consider the operator

$$\widetilde{K}_2 = M_2^{-1}K_2. \tag{4}$$

Simple computations show that

$$\widetilde{K}_2 = (A_0 I + A_1 U + A_2 U^2 + \cdots + A_m U^m + A_{m+1} U^{m+1} + \cdots + A_{2m} U^{2m})P_+ + P_-, \tag{5}$$

where

$$A_0 = \begin{pmatrix} -a^{-1} & a^{-1}a_0 \\ -a^{-1}\overline{a_0} & a^{-1}|a_0|^2 - \overline{a} \end{pmatrix}, \tag{6}$$

$$A_1 = \begin{pmatrix} 0 & a^{-1}a_1 \\ -a^{-1}(\alpha)\overline{a_1} & a^{-1}\overline{a_0}a_1 + a^{-1}(\alpha)\overline{a_1}a_0(\alpha) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & a^{-1}a_2 \\ -a^{-1}(\alpha_2)\overline{a_2} & a^{-1}\overline{a_0}a_2 + a^{-1}(\alpha)\overline{a_1}a_1(\alpha) + a^{-1}(\alpha_2)\overline{a_2}a_0(\alpha_2) \end{pmatrix},$$

$$\ldots,$$

$$A_m = \begin{pmatrix} 0 & a^{-1}a_m \\ -a^{-1}(\alpha_m)\overline{a_m} & a^{-1}\overline{a_0}a_m + a^{-1}(\alpha)\overline{a_1}a_{m-1}(\alpha) + \cdots + a^{-1}(\alpha_m)\overline{a_m}a_0(\alpha_m) \end{pmatrix},$$

$$A_{m+1} = \begin{pmatrix} 0 & 0 \\ 0 & a^{-1}(\alpha)\overline{a_1}a_m(\alpha) + a^{-1}(\alpha_2)\overline{a_2}a_{m-1}(\alpha_2) + \cdots + a^{-1}(\alpha_m)\overline{a_m}a_1(\alpha_m) \end{pmatrix},$$

$$\ldots,$$

$$A_{2m} = \begin{pmatrix} 0 & 0 \\ 0 & a^{-1}(\alpha_m)\overline{a_m}a_m(\alpha_m) \end{pmatrix}.$$

Taking into account Proposition 2.1 and (4) we have

Proposition 2.2. *Let* $\widetilde{K}_2 : \widetilde{L}_2^2(\mathbb{R}) \to \widetilde{L}_2^2(\mathbb{R})$ *be the singular integral operator with shift defined by* (5)*, then*

$$n = \frac{1}{2} \dim \ker \widetilde{K}_2.$$

Let E_n denote the $(n \times n)$ identity matrix and, for simplicity, $E \equiv E_2$.

Proposition 2.3. *Let* $K_3 : \widetilde{L}_2^{4m}(\mathbb{R}) \to \widetilde{L}_2^{4m}(\mathbb{R})$ *be the paired operator with shift*

$$K_3 = (B_0 I + B_1 U)P_+ + P_-, \tag{7}$$

where B_0 *and* B_1 *are the* $(4m \times 4m)$ *matrix functions*

$$B_0 = \begin{pmatrix} A_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & E & 0 & \cdots & & 0 \\ 0 & 0 & E & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & 0 & 0 \\ 0 & & & \ddots & E & 0 \\ 0 & 0 & & \cdots & 0 & E \end{pmatrix}$$

$$B_1 = \begin{pmatrix} A_1 & A_2 & \cdots & A_{2m-2} & A_{2m-1} & A_{2m} \\ -E & 0 & \cdots & 0 & 0 & 0 \\ 0 & -E & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & -E & 0 & 0 \\ 0 & 0 & \cdots & 0 & -E & 0 \end{pmatrix}; \tag{8}$$

then

$$n = \frac{1}{2} \dim \ker K_3.$$

Proof. Let $N : \widetilde{L}_2^{4m}(\mathbb{R}) \to \widetilde{L}_2^{4m}(\mathbb{R})$ be the invertible operator

$$N = \begin{pmatrix} I & 0 & \cdots & 0 & 0 & 0 \\ UP_+ & I & \ddots & \vdots & \vdots & \vdots \\ U^2P_+ & UP_+ & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & I & 0 & 0 \\ U^{2m-2}P_+ & U^{2m-3}P_+ & \cdots & UP_+ & I & 0 \\ U^{2m-1}P_+ & U^{2m-2}P_+ & \cdots & U^2P_+ & UP_+ & I \end{pmatrix},$$

with $I, U^k P_+ : \widetilde{L}_2^2(\mathbb{R}) \to \widetilde{L}_2^2(\mathbb{R})$, $k = \overline{1, 2m-1}$.

We obtain that

$$K_3N = \begin{pmatrix} \widetilde{K}_2 & F_1 & F_2 & \cdots & F_{2m-2} & F_{2m-1} \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & 0 & 0 \\ 0 & 0 & & \ddots & I & 0 \\ 0 & 0 & & \cdots & 0 & I \end{pmatrix},$$

where $F_1 = (A_2U + \cdots + A_{2m}U^{2m-1})P_+$, $F_2 = (A_3U + \cdots + A_{2m}U^{2m-2})P_+$, $\ldots$, $F_{2m-2} = (A_{2m-1}U + A_{2m}U^2)P_+$, $F_{2m-1} = A_{2m}UP_+$.

Thus

$$n = \frac{1}{2}\dim\ker \widetilde{K}_2 = \frac{1}{2}\dim\ker K_3. \qquad \square$$

Now we analyze the matrix A_0,

$$A_0 = \begin{pmatrix} -a^{-1} & a^{-1}a_0 \\ -a^{-1}\overline{a_0} & a^{-1}|a_0|^2 - \overline{a} \end{pmatrix},$$

in more detail. Note that $\det A_0(t) \neq 0$ for all $t \in \overset{\circ}{\mathbb{R}}$. It is known that (see, for instance, [18]; see also [2], [3] and [5]) the non-singular continuous matrix function A_0 admits the following (right) factorization in $L_2^{2\times 2}(\mathbb{R})$

$$A_0 = A_-\Lambda A_+, \tag{9}$$

where

$$(t-i)^{-1}A_-^{\pm 1} \in \left[\widehat{L}_2^-(\mathbb{R})\right]^{2\times 2}, (t+i)^{-1}A_+^{\pm 1} \in \left[\widehat{L}_2^+(\mathbb{R})\right]^{2\times 2},$$

$$\Lambda = \operatorname{diag}(\theta^{\varkappa_1}, \theta^{\varkappa_2}), \quad \theta(t) = \frac{t-i}{t+i},$$

$\varkappa_1, \varkappa_2 \in \mathbb{Z}$, with $\varkappa_1 \geq \varkappa_2$, $\widehat{L}_2^{\pm}$ are the spaces of the Fourier transforms of the functions of $L_2^{\pm}$, respectively, and $L_2^+ = P_+L_2$, $L_2^- = P_-L_2 \oplus \mathbb{C}$. The integers $\varkappa_1, \varkappa_2$ are uniquely defined by the matrix function A_0 and are called its partial indices. It is assumed that

$$(t-i)^{-1}A_-^{\pm 1}, (t+i)^{-1}A_+^{\pm 1} \in C^{2\times 2}(\mathbb{R}).$$

Proposition 2.4. *Let $a \in C(\overset{\circ}{\mathbb{R}})$, $a(t) \neq 0$, for all $t \in \overset{\circ}{\mathbb{R}}$, and let*

$$a = a_-\theta^{k_a}a_+, \quad k_a = \frac{1}{2\pi}\{\arg a(t)\}_{t\in\overset{\circ}{\mathbb{R}}},$$

be a factorization of a in $L_2(\mathbb{R})$. Then the partial indices of the matrix A_0 are

$$\varkappa_1 = -k_a + k, \quad \varkappa_2 = -k_a - k,$$

where

$$k = \dim\ker(I - P_-u_-P_+\overline{u_-}P_-), \tag{10}$$

and $u_- := P_-u$, $u := (a_0)_-(\overline{a_-}a_+)^{-1}$, $(a_0)_- := P_-(a_0)$.

Proof. The matrix A_0 can be written as the product

$$A_0 = \theta^{-k_a} B_- M B_+,$$

where

$$B_- = \begin{pmatrix} 1 & 0 \\ \overline{(a_0)_+} & 1 \end{pmatrix} \begin{pmatrix} a_-^{-1} & 0 \\ 0 & \overline{a_+} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \overline{u_+} & 1 \end{pmatrix},$$
$$B_+ = \begin{pmatrix} 1 & u_+ \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_+^{-1} & 0 \\ 0 & \overline{a_-} \end{pmatrix} \begin{pmatrix} -1 & (a_0)_+ \\ 0 & 1 \end{pmatrix},$$
$$M = \begin{pmatrix} 1 & u_- \\ \overline{u_-} & |u_-|^2 - 1 \end{pmatrix},$$

and $(a_0)_+ := P_+(a_0)$, $u_+ := P_+ u$.

The matrix M admits the following factorization in $L_2^{2\times 2}(\mathbb{R})$ (see [18])

$$M = M_- \operatorname{diag}(\theta^k, \theta^{-k}) M_+.$$

Then a factorization of the matrix A_0 is

$$A_0 = B_- M_- \operatorname{diag}(\theta^{-k_a+k}, \theta^{-k_a-k}) M_+ B_+,$$

where $\varkappa_1 = -k_a + k$, $\varkappa_2 = -k_a - k$ are its partial indices. □

Proposition 2.5. *Let $a \in C(\overset{\circ}{\mathbb{R}})$, $a(t) \neq 0$, for all $t \in \overset{\circ}{\mathbb{R}}$, B_1 be the matrix function defined by* (8)*, $A_\pm$ and $\varkappa_{1,2}$ be the external factors and the partial indices, respectively, of the factorization of the matrix A_0 in $L_2^{2\times 2}(\mathbb{R})$. Then*

$$n \leq \frac{1}{2}(\dim\ker K - 2\varkappa_1^- - 2\varkappa_2^-),$$

where $K : \widetilde{L}_2^{4m}(\mathbb{R}) \to \widetilde{L}_2^{4m}(\mathbb{R})$ is the paired operator

$$K = (I + FU)P_+ + P_-, \tag{11}$$

F is the $(4m \times 4m)$ matrix function

$$F = \operatorname{diag}(\Lambda_-^{-1} A_-^{-1}, E_{4m-2}) B_1 \operatorname{diag}(A_+^{-1}(\alpha)\Lambda_+^{-1}(\alpha), E_{4m-2}), \tag{12}$$

$$\Lambda_\pm : \Lambda = \Lambda_- \Lambda_+, \quad \Lambda_\pm = \operatorname{diag}(\theta^{\varkappa_1^\pm}, \theta^{\varkappa_2^\pm}),$$

Λ is the central factor of the factorization of the matrix A_0 and

$$\varkappa_j^\pm = \frac{1}{2}(\varkappa_j \pm |\varkappa_j|), \quad j = 1, 2,$$

with

$$\varkappa_j = \varkappa_j^+ + \varkappa_j^-, \quad j = 1, 2,$$

the partial indices of A_0.

Proof. The operator K_3 defined by (7) admits the factorization

$$K_3 = \operatorname{diag}(A_-, E_{4m-2})K_\Lambda[\operatorname{diag}(A_+, E_{4m-2})P_+ + \operatorname{diag}(A_-^{-1}, E_{4m-2})P_-], \quad (13)$$

where

$$K_\Lambda = \left[\operatorname{diag}(\Lambda, E_{4m-2})I + \widetilde{F}U\right] P_+ + P_-,$$

with

$$\widetilde{F} = \operatorname{diag}(A_-^{-1}, E_{4m-2})B_1 \operatorname{diag}(A_+^{-1}(\alpha), E_{4m-2}).$$

The following equalities hold

$$K_\Lambda K_- = \operatorname{diag}(\Lambda_-, E_{4m-2})\widetilde{K}, \quad (14)$$

$$\widetilde{K} = KK_+, \quad (15)$$

where $K_\pm$ are the left invertible operators

$$K_- = P_+ + \operatorname{diag}(\Lambda_-, E_{4m-2})P_-,$$
$$K_+ = \operatorname{diag}(\Lambda_+, E_{4m-2})P_+ + P_-,$$
$$\widetilde{K} = \left[\operatorname{diag}(\Lambda_+, E_{4m-2})I + \operatorname{diag}(\Lambda_-^{-1}, E_{4m-2})\widetilde{F}U\right] P_+ + P_-,$$

and K is the operator defined above.

It follows from (14) and (15) that

$$\dim\ker K_\Lambda \le \dim\ker \widetilde{K} + \dim\operatorname{coker} K_-,$$

and

$$\dim\ker \widetilde{K} \le \dim\ker K.$$

Finally, taking into account the invertibility in $\widetilde{L}_2^{4m}(\mathbb{R})$ of the first and the third operators in (13), Proposition 2.3 and the relation (see [20])

$$\dim\operatorname{coker} K_- = -2\varkappa_1^- - 2\varkappa_2^-,$$

we obtain

$$n = \frac{1}{2}\dim\ker K_\Lambda \le \frac{1}{2}(\dim\ker K - 2\varkappa_1^- - 2\varkappa_2^-). \qquad \square$$

Thus, it remains to estimate $\dim\ker K$. To do this we need a few more facts. As usual, let $\mathbb{T}_+$ denote the interior of the unit disk, $\sigma(g)$, $\rho(g)$ and $\|g\|_2$, denote the spectrum, the spectral radius and the spectral norm of a matrix $g \in \mathbb{C}^{n\times n}$, respectively. Now we will make use of some results from [12]:

Lemma 2.1 ([12]). *For every continuous matrix function $h \in C^{n\times n}(\overset{\circ}{\mathbb{R}})$ such that*

$$\sigma[h(\infty)] \subset \mathbb{T}_+,$$

there exist an induced matrix norm $\|.\|_0$ and a rational matrix r such that

$$\max_{t\in\overset{\circ}{\mathbb{R}}} \left\|r(t)h(t)r^{-1}(t+\mu)\right\|_0 < 1 \quad (16)$$

and

$$P_+ r^{\pm 1} P_+ = r^{\pm 1} P_+. \quad (17)$$

Note that the rational matrix r can and must be chosen such that the condition (16), with the spectral norm instead of the induced norm $\|.\|_0$, holds; unfortunately this fact was not duly emphasized in [12].

The rational matrices r satisfying the conditions (16) and (17) have the elements of the form $r_{i,j}(t) = p_{i,j}\left(\frac{t-i}{t+i}\right)$, where $p_{i,j}$ is a polynomial, $i, j = \overline{1,n}$ (i.e., $r_{i,j}$ is a rational function whose zeros lie in the lower half-plane and has no poles). Let R_h denote the set of all such rational matrices r,

$$l_1(r) = \sum_{i=1}^{n} \max_{j=\overline{1,n}} l_{i,j},$$

where $l_{i,j}$ is the degree of the element $r_{i,j}(t) = p_{i,j}\left(\frac{t-i}{t+i}\right)$ of the rational matrix r, and

$$l(h) = \min_{r\in R_h} \{l_1(r)\}. \tag{18}$$

Lemma 2.2. [12] *Let $T = (I - gU)P_+ + P_- : L_2^n(\mathbb{R}) \to L_2^n(\mathbb{R})$, where the matrix function g satisfies the conditions of the Lemma* 2.1*. Then the estimate*

$$\dim\ker T \le l(g),$$

holds.

Proposition 2.6. *Let K be the operator defined by* (11) *and $a \in C(\overset{\circ}{\mathbb{R}})$, $a(t) \neq 0$, for all $t \in \overset{\circ}{\mathbb{R}}$. Then*

$$\dim\ker K \le 2l(F),$$

where $l(F)$ is the number defined by (18) *for the matrix F.*

Proof. Taking into account Lemmas 2.1 and 2.2, it suffices to show that

$$\sigma[F(\infty)] \subset \mathbb{T}_+.$$

From the factorization $A_0 = A_-\Lambda A_+$ of the matrix function A_0, we have

$$A_0(\infty) = A_-(\infty)A_+(\infty),$$

so

$$A_+^{-1}(\infty) = A_0^{-1}(\infty)A_-(\infty).$$

Now using (12), we can write

$$F(\infty) = \operatorname{diag}(A_-^{-1}(\infty), E_{4m-2})B_1(\infty)\operatorname{diag}(A_+^{-1}(\infty), E_{4m-2})$$

and so

$$F(\infty) = \operatorname{diag}(A_-^{-1}(\infty), E_{4m-2})B_1(\infty)\operatorname{diag}(A_0^{-1}(\infty), E_{4m-2})\operatorname{diag}(A_-(\infty), E_{4m-2}),$$

which means that the matrices $F(\infty)$ and $B_1(\infty)\operatorname{diag}(A_0^{-1}(\infty), E_{4m-2})$ are similar.

We show that all eigenvalues of the matrix

$$B_1(\infty)\mathrm{diag}\,(A_0^{-1}(\infty), E_{4m-2})$$
$$= \begin{pmatrix} A_1(\infty)A_0^{-1}(\infty) & A_2(\infty) & A_3(\infty) & \cdots & A_{2m-1}(\infty) & A_{2m}(\infty) \\ -A_0^{-1}(\infty) & 0 & 0 & \cdots & 0 & 0 \\ 0 & -E & 0 & \cdots & 0 & 0 \\ 0 & 0 & -E & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & -E & 0 \end{pmatrix},$$

are equal to 0.

Let B be the matrix

$$B = \begin{pmatrix} A_1(\infty)A_0^{-1}(\infty) - zE & A_2(\infty) & A_3(\infty) & \cdots & A_{2m-1}(\infty) & A_{2m}(\infty) \\ -A_0^{-1}(\infty) & -zE & 0 & \cdots & 0 & 0 \\ 0 & -E & -zE & \cdots & 0 & 0 \\ 0 & 0 & -E & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & -zE & 0 \\ 0 & 0 & 0 & \cdots & -E & -zE \end{pmatrix},$$

where $z \in \mathbb{C}$. We have that

$$B = G + H,$$

where

$$G = \begin{pmatrix} -zE & 0 & 0 & \cdots & 0 & 0 \\ -A_0^{-1}(\infty) & -zE & 0 & \cdots & 0 & 0 \\ 0 & -E & -zE & \cdots & 0 & 0 \\ 0 & 0 & -E & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & -zE & 0 \\ 0 & 0 & 0 & \cdots & -E & -zE \end{pmatrix},$$

$$H = \begin{pmatrix} A_1(\infty)A_0^{-1}(\infty) & A_2(\infty) & A_3(\infty) & \cdots & A_{2m-1}(\infty) & A_{2m}(\infty) \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & & \vdots & \vdots \\ \vdots & & & & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Moreover

$$G^{-1} = \begin{pmatrix} -z^{-1}E & 0 & \cdots & 0 & 0 & 0 \\ z^{-2}A_0^{-1}(\infty) & -z^{-1}E & \ddots & \vdots & \vdots & \vdots \\ -z^{-3}A_0^{-1}(\infty) & z^{-2}E & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & -z^{-1}E & 0 & 0 \\ -z^{-2m+1}A_0^{-1}(\infty) & z^{-2m+2}E & \ddots & z^{-2}E & -z^{-1}E & 0 \\ z^{-2m}A_0^{-1}(\infty) & -z^{-2m+1}E & \cdots & -z^{-3}E & z^{-2}E & -z^{-1}E \end{pmatrix}.$$

Furthermore, we have

$$H = XEY,$$

where

$$X = \begin{pmatrix} E & 0 & \cdots & 0 \end{pmatrix}^T$$

and

$$Y = \begin{pmatrix} A_1(\infty)A_0^{-1}(\infty) & A_2(\infty) & A_3(\infty) & \cdots & A_{2m-1}(\infty) & A_{2m}(\infty) \end{pmatrix}$$

are $(4m \times 2)$ and $(2 \times 4m)$ matrices, respectively.

Then

$$B = G + XEY.$$

We compute the (2×2) matrix,

$$E + YG^{-1}X = \Big(A_0(\infty) - z^{-1}A_1(\infty) + z^{-2}A_2(\infty) - \cdots$$
$$\cdots - z^{-2m+1}A_{2m+1}(\infty) + z^{-2m}A_{2m}(\infty)\Big)A_0^{-1}(\infty)$$

and show that its determinant is equal to one for all $z \neq 0$. Thus the matrix $E + YG^{-1}X$ is invertible, and so is the matrix B (see, for instance, Chapter 0.7.4 in [6]), with

$$B^{-1} = G^{-1} - G^{-1}X(E + YG^{-1}X)^{-1}YG^{-1}.$$

It follows that the resolvent set of the matrix $B_1(\infty)\mathrm{diag}\,(A_0^{-1}(\infty), E_{4m-2})$ is $\mathbb{C} \setminus \{0\}$, so the spectrum is

$$\sigma[B_1(\infty)\mathrm{diag}\,(A_0^{-1}(\infty), E_{4m-2})] = \{0\}.$$

Thus

$$\sigma[F(\infty)] = \{0\}. \qquad \square$$

Finally, Propositions 2.5 and 2.6 allow us to establish our main result.

Theorem 2.1. *Let* $a \in C(\mathring{\mathbb{R}})$, $a(t) \neq 0$, *for all* $t \in \mathring{\mathbb{R}}$, *and let* $k_a = \mathrm{ind}\, a$, k *be the number defined by* (10), F *be the matrix function defined by* (12), $l(F)$ *be the number defined by* (18) *for the matrix* F, *and* n *be the number of linearly independent solutions of the problem* (1). *Then the estimate*

$$n \leq l(F) + \max(k_a - k, 0) + \max(k_a + k, 0), \tag{19}$$

holds.

Remark. By Proposition 2.4, the partial indices of the matrix A_0 are $\varkappa_1 = -k_a + k$ and $\varkappa_2 = -k_a - k$. Therefore, estimate (19) can be written as

$$n \le l(F) + \max(-\varkappa_1, 0) + \max(-\varkappa_2, 0). \tag{20}$$

3. A special case with analytic coefficients

Let us consider the problem

$$\varphi_+ = a\varphi_- + a_0\overline{\varphi_-} + a_1\overline{\varphi_-(\alpha)} + a_2\overline{\varphi_-(\alpha_2)} + \cdots + a_m\overline{\varphi_-(\alpha_m)}, \tag{21}$$

in the case when $a_1, \dots, a_m \in C(\overset{\circ}{\mathbb{R}})$ have analytic continuation into the upper half-plane.

Let $N_2 : \widetilde{L}_2^2(\mathbb{R}) \to \widetilde{L}_2^2(\mathbb{R})$ be the invertible operator

$$N_2 = \begin{pmatrix} 1 & A - a_0 \\ 0 & 1 \end{pmatrix} P_+ + \begin{pmatrix} 1 & 0 \\ \widetilde{A} - \overline{a_0} & 1 \end{pmatrix} P_-.$$

Recall that $A = \sum\limits_{j=0}^{m} a_j U^j$ and $\widetilde{A} = \sum\limits_{j=0}^{m} \overline{a_j} U^j$.

We define the operator $T : \widetilde{L}_2^2(\mathbb{R}) \to \widetilde{L}_2^2(\mathbb{R})$ by

$$T = K_2 N_2,$$

where K_2 is the operator defined by (3). It is easily seen that

$$T = \begin{pmatrix} -1 & a_0 \\ 0 & \overline{a} \end{pmatrix} P_+ + \begin{pmatrix} a & 0 \\ \overline{a_0} & -1 \end{pmatrix} P_-.$$

The number of linearly independent solutions of the problem (21) is given by

$$n = \frac{1}{2} \dim \ker T.$$

Notice that

$$\begin{pmatrix} -1 & a_0 \\ 0 & \overline{a} \end{pmatrix}^{-1} \begin{pmatrix} a & 0 \\ \overline{a_0} & -1 \end{pmatrix} = A_0^{-1},$$

where A_0 is the matrix function defined by (6). From (9) the matrix A_0^{-1} admits the left factorization in $L_2^{2\times 2}(\mathbb{R})$

$$A_0^{-1} = A_+^{-1} \Lambda^{-1} A_-^{-1},$$

where $\Lambda^{-1} = \operatorname{diag}(\theta^{-\varkappa_1}, \theta^{-\varkappa_2})$. It is known (see [18]) that

$$\dim \ker T = 2[\max(-\varkappa_1, 0) + \max(-\varkappa_2, 0)].$$

Thus we have

Proposition 3.1. *Let $a \in C(\overset{\circ}{\mathbb{R}})$, $a(t) \neq 0$, for all $t \in \overset{\circ}{\mathbb{R}}$, and let $\varkappa_1$, $\varkappa_2$, be the partial indices of the matrix A_0. Then the number of linearly independent solutions of the problem* (21) *is*

$$n = \max(-\varkappa_1, 0) + \max(-\varkappa_2, 0). \tag{22}$$

On the other hand, the matrix function A_0 admits the right factorization (9),

$$A_0 = A_- \Lambda A_+,$$

where $\Lambda = \operatorname{diag}(\theta^{\varkappa_1}, \theta^{\varkappa_2})$. Theorem 2.1 yields the estimate (20) which is obviously consistent with the equality (22).

Example. Consider the following boundary value problem

$$\varphi_+ = \theta\varphi_- + \overline{\varphi_-} + 10^{-3}\overline{\varphi_-(\alpha)}. \tag{23}$$

Note that the two pairs of functions

$$\psi_+ = \frac{2}{t+i} + \frac{10^{-3}}{t+\mu+i}, \quad \psi_- = \frac{1}{t-i},$$

and

$$\phi_+ = \frac{-10^{-3}i}{t+\mu+i}, \qquad \phi_- = \frac{i}{t-i},$$

are linearly independent solutions of the problem (23).

The matrix A_0 associated with the problem (23) has the form

$$\begin{aligned} A_0 &= \begin{pmatrix} -\theta^{-1} & \theta^{-1} \\ -\theta^{-1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \theta^{-1} & 0 \\ 0 & \theta^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \end{aligned}$$

and the corresponding (4×4) matrix F in (12) is

$$F = \begin{pmatrix} 10^{-3}\theta\theta^{-1}(\alpha) & 10^{-3}(-1 - \theta\theta^{-1}(\alpha)) & 0 & -10^{-6}\theta\theta^{-1}(\alpha) \\ 10^{-3}\theta\theta^{-1}(\alpha) & -10^{-3}\theta\theta^{-1}(\alpha) & 0 & -10^{-6}\theta\theta^{-1}(\alpha) \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

If D is the diagonal matrix

$$D = \operatorname{diag}(3^{-1}, 2^{-1}, 10^{-3}, 10^{-3}),$$

then one can easily check that

$$\left\| DF(\infty)D^{-1} \right\|_2 < 1,$$

and

$$\max_{t \in \overset{0}{\mathbb{R}}} \left\| DF(t)D^{-1} \right\|_2 < 1.$$

Therefore, $r(t) = D$, that implies $l(F) = 0$.

Now, from (22) and (20), we get

$$2 = n \le 2.$$

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On the Hyperderivatives of Dirac-hyperholomorphic Functions of Clifford Analysis

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Abstract. In the context of Clifford analysis, considering the Cauchy-Riemann and Dirac operators one has that any Dirac-hyperholomorphic function is also Cauchy-Riemann-hyperholomorphic, but its hyperderivative in the Cauchy-Riemann sense is always zero, so these functions can be thought of as "constants" for the Cauchy-Riemann operator. It turns out that it is possible to give another kind of hyperderivatives "consistent" with the Dirac operator, but there are several of them. We focus in detail on one of these hyperderivatives and develop also the notion of $(n-1)$-dimensional directional hyperderivative along a hyperplane. As in the previous works, an application to the Cliffordian-Cauchy-type integral proves to be instructive.

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1. Introduction

1.1. Algebraic preliminaries

1.1.1. We will use the common notation $C\ell_{0,m}$ to denote the real Clifford algebra with imaginary units $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ with negative signature; denoting $\mathbf{e}_0 = 1$ the real unit, any Clifford number is of the form $a = \sum_A a_A \mathbf{e}_A$, where $\mathbf{e}_A := \mathbf{e}_{h_1}\mathbf{e}_{h_2}\cdots\mathbf{e}_{h_r}$, $1 \leq h_1, < \cdots < h_r \leq m, \mathbf{e}_\emptyset = \mathbf{e}_0 = 1$ and $a_A \in \mathbb{R}$. The conjugate of a is defined by $\overline{a} := \sum_A a_A \,\overline{\mathbf{e}}_A$, with $\overline{\mathbf{e}}_A := \overline{\mathbf{e}}_{h_r}\overline{\mathbf{e}}_{h_{r-1}}\cdots\overline{\mathbf{e}}_{h_1}$; $\overline{\mathbf{e}}_k := -\mathbf{e}_k \,(k = 1, \ldots, m)$, $\overline{\mathbf{e}}_0 = \mathbf{e}_0 = 1$.

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The real-linear subspaces

$$C\ell_{0,m}^{(k)} := \Big\{ a \in C\ell_{0,m} \,|\, a = \sum_{|A|=k} a_A e_A \Big\}, \quad k \in \{0,1,\dots,m\},$$

induce a decomposition of the Clifford algebra $C\ell_{0,m}$ given by

$$C\ell_{0,m} = C\ell_{0,m}^{+} \oplus C\ell_{0,m}^{-}\,, \tag{1.1}$$

where $C\ell_{0,m}^{+} := \oplus_{k\,\text{even}}\, C\ell_{0,m}^{(k)}$ and $C\ell_{0,m}^{-} := \oplus_{k\,\text{odd}}\, C\ell_{0,m}^{(k)}$. It is seen at once that $C\ell_{0,m}^{+}$ is a subalgebra and it is called the *even subalgebra*. Thus any $y \in C\ell_{0,m}$ has a unique representation $y = y^{+} + y^{-}$, with $y^{+} \in C\ell_{0,m}^{+}$ and $y^{-} \in C\ell_{0,m}^{-}$.

1.1.2. There is a decomposition of the Clifford algebra $C\ell_{0,m}$ that we shall use in what follows.

The generator of the real-linear subspace $C\ell_{0,m}^{(m)}$ is $\mathbf{e}_{\mathbb{N}_m} := \mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_m$, with $\mathbb{N}_m := \{1,2,\dots,m\}$ and it has the property: $\mathbf{e}_{\mathbb{N}_m}^2 = 1$ or -1. The elements of the set $\mathbf{e}_{\mathbb{N}_m}\mathbb{R}$ are called *pseudo-scalars*. On the other hand any $y^{-} \in C\ell_{0,m}^{-}$ is written as

$$y^{-} = \sum_{\ell=1}^{m} y_\ell \mathbf{e}_\ell + \sum_{|A|=3} y_A \mathbf{e}_A + \cdots + \sum_{|A|=\beta-2} y_A \mathbf{e}_A + \sum_{|A|=\beta} y_A \mathbf{e}_A\,,$$

where

$$\beta = \begin{cases} m & \text{if } m \text{ is odd;} \\ m-1 & \text{if } m \text{ is even.} \end{cases}$$

In the case that m is an odd number, it is direct to see that

$$\mathbf{e}_{\mathbb{N}_m}^2 = \begin{cases} -1 & \text{if} \quad m \equiv 1 \bmod 4, \\ 1 & \text{if} \quad m \equiv 3 \bmod 4. \end{cases}$$

Thus, in this case, y^{-} is written as follows:

$$y^{-} = \Big(\pm \sum_{\ell=1}^{m} y_\ell \mathbf{e}_\ell \mathbf{e}_{\mathbb{N}_m} \pm \sum_{|A|=3} y_A \mathbf{e}_A \mathbf{e}_{\mathbb{N}_m} \pm \cdots \pm \sum_{|A|=\beta-2} y_A \mathbf{e}_A \mathbf{e}_{\mathbb{N}_m} + y_{\mathbb{N}_m} \Big) \mathbf{e}_{\mathbb{N}_m}\,,$$

where the sign depends on the class of congruence of $m \bmod 4$. Observe that, if $|\,A\,|$ is odd, then $\mathbf{e}_A\mathbf{e}_{\mathbb{N}_m} = \mathbf{e}_{A'}$, with $|\,A'\,|$ an even number, that is, every product $\mathbf{e}_A\mathbf{e}_{\mathbb{N}_m} \in C\ell_{0,m}^{+}$, hence

$$\pm \sum_{\ell=1}^{m} y_\ell \mathbf{e}_\ell \mathbf{e}_{\mathbb{N}_m} \pm \sum_{|A|=3} y_A \mathbf{e}_A \mathbf{e}_{\mathbb{N}_m} \pm \cdots \pm \sum_{|A|=\beta-2} y_A \mathbf{e}_A \mathbf{e}_{\mathbb{N}_m} + y_{\mathbb{N}_m} \in C\ell_{0,m}^{+}\,.$$

Considering now the case of m even, and taking $k \in \{1, 2, \ldots, m\}$, one has:

$$\begin{aligned} y^- &= \sum_{\ell=1}^{m} y_\ell \mathbf{e}_\ell + \sum_{|A|=3} y_A \mathbf{e}_A + \cdots + \sum_{|A|=m-3} y_A \mathbf{e}_A + \sum_{|A|=m-1} y_A \mathbf{e}_A \\ &= -\sum_{\ell=1}^{m} y_\ell \mathbf{e}_\ell \, (\mathbf{e}_k)^2 - \sum_{|A|=3} y_A \mathbf{e}_A \, (\mathbf{e}_k)^2 - \cdots \\ &\qquad \cdots - \sum_{|A|=m-3} y_A \mathbf{e}_A \, (\mathbf{e}_k)^2 - \sum_{|A|=m-1} y_A \mathbf{e}_A \, (\mathbf{e}_k)^2 \\ &= \Bigg(-\sum_{\ell=1}^{m} y_\ell \mathbf{e}_\ell \, \mathbf{e}_k - \sum_{|A|=3} y_A \mathbf{e}_A \, \mathbf{e}_k - \cdots \\ &\qquad \cdots - \sum_{|A|=m-3} y_A \mathbf{e}_A \, \mathbf{e}_k - \sum_{|A|=m-1} y_A \mathbf{e}_A \, \mathbf{e}_k \Bigg) \mathbf{e}_k \, . \end{aligned}$$

Again, if $\mid A \mid$ is odd, then $\mathbf{e}_A \mathbf{e}_k = \mathbf{e}_{A'}$, with $\mid A' \mid$ always an even number, and every product $\mathbf{e}_A \mathbf{e}_{N_m} \in C\ell_{0,m}^+$, hence

$$-\sum_{\ell=1}^{m} y_\ell \mathbf{e}_\ell \, \mathbf{e}_k - \sum_{|A|=3} y_A \mathbf{e}_A \, \mathbf{e}_k - \cdots - \sum_{|A|=m-3} y_A \mathbf{e}_A \, \mathbf{e}_k - \sum_{|A|=m-1} y_A \mathbf{e}_A \, \mathbf{e}_k \in C\ell_{0,m}^+ .$$

With the above reasoning it is clear that the linear subspace $C\ell_{0,m}^-$ can be written (not in a unique way) as:

$$C\ell_{0,m}^- = C\ell_{0,m}^+ \alpha \, . \tag{1.2}$$

with $\alpha = \mathbf{e}_{\mathbb{N}_m}$ or $\alpha = \mathbf{e}_k$ depending on if m is odd or even, but in any case α satisfies $\alpha^2 = 1$ or $\alpha^2 = -1$.

Thus from the decomposition (1.1) one has:

$$C\ell_{0,m} \cong C\ell_{0,m}^+ \oplus C\ell_{0,m}^+ \alpha \, . \tag{1.3}$$

It was proved in Section 1.7.2 in [1] that the subalgebra $C\ell_{0,m}^+$ is isomorphic to the algebra $C\ell_{0,m-1}$ thus we can say more about the decomposition of the Clifford algebra $C\ell_{0,m}$:

$$C\ell_{0,m} \cong C\ell_{0,m-1} \oplus C\ell_{0,m-1} \alpha \, . \tag{1.4}$$

1.1.3. One more algebraic fact. Fixing $\mathbf{e}_\ell$, with $\ell \in \{1, \ldots, m\}$, consider the $m-1$ bivectors $\mathbf{e}_{j,\ell} := \mathbf{e}_j \mathbf{e}_\ell$, with $j \in \{1, \ldots, m\}$ and $j \neq \ell$. Let $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$. An alternative Cliffordian representation of x is given by

$$x = x_\ell + \sum_{j \in \{1, \ldots, \ell-1, \ell+1, \ldots, m\}} x_j \, \mathbf{e}_{j,\ell} \, ,$$

hence any domain $\Omega_m \subset \mathbb{R}^m$ can be embedded in this other way into the Clifford algebra $C\ell_{0,m-1}$ which is seen inside the Clifford algebra $C\ell_{0,m}$.

1.2. Complex and hypercomplex antecedents

1.2.1. In [9], Shapiro and Vasilevski obtained the correlation:

$$d(\sigma_x^{(2)} f(x)) = \frac{1}{2} \left(\sigma_x^{(3)} \overline{\mathcal{D}}_F[f](x) - \overline{\sigma}_x^{(3)} \mathcal{D}_F[f](x) \right), \tag{1.5}$$

where $\sigma_x^{(3)}$ is the 3-form

$$\begin{aligned} \sigma_x^{(3)} &:= dx_1 \wedge dx_2 \wedge dx_3 - \mathbf{i}\, dx_0 \wedge dx_2 \wedge dx_3 \\ &\quad + \mathbf{j}\, dx_0 \wedge dx_1 \wedge dx_3 - \mathbf{k}\, dx_0 \wedge dx_1 \wedge dx_2 \\ &=: d\widehat{x}_0 - \mathbf{i}\, d\widehat{x}_1 + \mathbf{j}\, d\widehat{x}_2 - \mathbf{k}\, d\widehat{x}_3 , \end{aligned}$$

$\sigma_x^{(2)}$ is the 2-form

$$\begin{aligned} \sigma_x^{(2)} &:= \mathbf{i}\, dx_2 \wedge dx_3 - \mathbf{j}\, dx_1 \wedge dx_3 + \mathbf{k}\, dx_1 \wedge dx_2 \\ &=: \mathbf{i}\, d\widehat{x}_{1,0} - \mathbf{j}\, d\widehat{x}_{2,0} + \mathbf{k}\, d\widehat{x}_{3,0} , \end{aligned}$$

and

$$\mathcal{D}_F := \sum_{\ell=0}^{3} \mathbf{e}_\ell \frac{\partial}{\partial x_\ell}, \quad \overline{\mathcal{D}}_F := \sum_{\ell=0}^{3} \overline{\mathbf{e}}_\ell \frac{\partial}{\partial x_\ell}$$

are the Fueter operator and its (quaternionic) conjugate.

1.2.2. The equality (1.5) is a deep structural analog of its complex analysis antecedent

$$dg(z^0) = \frac{\partial g}{\partial z}(z^0)\, dz + \frac{\partial g}{\partial \overline{z}}(z^0)\, d\overline{z}. \tag{1.6}$$

In the latter, when g is a holomorphic function, then its complex derivative $g'(z^0)$ coincides with its "formal" derivative $\frac{\partial g}{\partial z}(z^0)$. Hence the analogy between (1.5) and (1.6) allows us to conclude that if a quaternionic function f is hyperholomorphic ($f \in \ker \mathcal{D}_F$) then $\frac{1}{2}\overline{\mathcal{D}}_F[f](x^0)$ is a highly probable candidate for being an "adequate quaternionic hyperderivative" of the hyperholomorphic function f. The paper [7] justifies the idea; it works with the notion of the hyperderivative $'f(x^0)$ of a Fueter-hyperholomorphic function as the limit of a specific quotient, concluding that

$$'f(x^0) = \frac{1}{2} \overline{\mathcal{D}}_F[f](x^0). \tag{1.7}$$

In others words recalling that in the complex case the derivative of a holomorphic function is the proportionality coefficient between the differentials of the function and of the independent variable: $dg = g'(z^0)\, dz$, we conclude that the hyperderivative $'f(x^0)$ is the proportionality coefficient between the two differential forms:

$$d(\sigma_{x^0}^{(2)} f(x^0)) = \frac{1}{2} \sigma_{x^0}^{(3)} \overline{\mathcal{D}}_F[f](x^0).$$

1.2.3. In [2], see also [5] the authors followed the above ideas in the context of real Clifford algebras $C\ell_{0,m}$. Let $\mathcal{D}_{CR} := \sum_{\ell=0}^{m} \mathbf{e}_\ell \frac{\partial}{\partial x_\ell}$ be the Cauchy-Riemann operator acting on $\mathcal{C}^1(\Omega \subset \mathbb{R}^{m+1}; \mathcal{C}\ell_{0,m})$; the (Cliffordian) Cauchy-Riemann-hyperholomorphic functions are null solutions of $\mathcal{D}_{CR}$. In analogy to (1.5) and (1.6), for $\mathcal{C}\ell_{0,m}$-valued functions of class $\mathcal{C}^1$ there holds:

$$d(\tau_x\, f(x)) = \frac{1}{2}\left(\overline{\sigma}_x \mathcal{D}_{CR}[f](x) - \sigma_x \overline{\mathcal{D}}_{CR}[f](x)\right) , \tag{1.8}$$

where

$$\sigma_x := d\hat{x}_0 - \mathbf{e}_1 d\hat{x}_1 + \cdots + (-1)^m \mathbf{e}_m d\hat{x}_m , \tag{1.9}$$

and

$$\tau_x = -\mathbf{e}_1 d\hat{x}_{0,1} + \mathbf{e}_2 d\hat{x}_{0,2} + \cdots + (-1)^m \mathbf{e}_m d\hat{x}_{0,m} , \tag{1.10}$$

with

$$d\widehat{x}_\ell := dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{\ell-1} \wedge dx_{\ell+1} \wedge \cdots \wedge dx_m ,$$

(i.e., $d\widehat{x}_\ell$ is obtained from the volume differential form $dV = dx_0 \wedge \cdots \wedge dx_m$, omitting the factor dx_ℓ), and $d\widehat{x}_{s,t}$ is obtained from $d\widehat{x}_s$ omitting also dx_t.

Similarly to what is written in Subsection 1.2.1 the Cliffordian hyperderivative is defined as the proportionality coefficient between the differential forms σ_x and τ_x, and it also turns out to be the limit of an appropriate quotient; besides, the hyperderivative coincides up to a real factor, as in (1.7), with the conjugate Cauchy-Riemann operator:

$$'f(x) = -\frac{1}{2}\overline{\mathcal{D}}_{CR}[f](x) .$$

1.2.4. The Dirac operator

$$\mathcal{D}_{\mathrm{Dir}} := \sum_{\ell=1}^{m} \mathbf{e}_\ell \frac{\partial}{\partial x_\ell} , \tag{1.11}$$

which acts on functions $f : \Omega \subset \mathbb{R}^m \to C\ell_{0,m}$ of class C^1, is related with the Cauchy-Riemann operator as follows: given f as before, define $\widetilde{f} : \mathbb{R} \times \Omega \subset \mathbb{R}^{m+1} \to C\ell_{0,m}$ such that

$$\widetilde{f}(x_0, x_1, x_2, \ldots, x_m) := f(x_1, x_2, \ldots, x_m) \text{ for any } x_0.$$

Now, assuming that f is Dirac-hyperholomorphic (that is, f is a null-solution of the Dirac operator), one has:

$$\begin{aligned} &\mathcal{D}_{CR}[\widetilde{f}](x_0, x_1, x_2, \ldots, x_m) \\ &\quad = \frac{\partial \widetilde{f}}{\partial x_0}(x_0, x_1, x_2, \ldots, x_m) + \sum_{\ell=1}^{m} \mathbf{e}_\ell \frac{\partial}{\partial x_\ell}[\widetilde{f}](x_0, x_1, x_2, \ldots, x_m) \\ &\quad = \sum_{\ell=1}^{m} \mathbf{e}_\ell \frac{\partial}{\partial x_\ell}[f](x_1, x_2, \ldots, x_m) = \mathcal{D}_{\mathrm{Dir}}[f](x_1, x_2, \ldots, x_m) = 0 , \end{aligned}$$

thus $\widetilde{f}$ is Cauchy-Riemann-hyperholomorphic in this specific domain $\mathbb{R} \times \Omega$ (of course, the "cylinder" can be of a finite height, and there are other ways of "inflating" the domain Ω). It is common to say that f is Cauchy-Riemann-hyperholomorphic as well, and to write the Cauchy-Riemann operator (for these specific domains and for these specific functions) as

$$\mathcal{D}_{CR} = \sum_{\ell=0}^{m} \mathbf{e}_\ell \frac{\partial}{\partial x_\ell} = \frac{\partial}{\partial x_0} + \mathcal{D}_{\mathrm{Dir}} \,.$$

What is more, the left-hyperderivative of $f \in \ker \mathcal{D}_{\mathrm{Dir}}$ at the point $x^0 = (x_1^0, x_2^0, \ldots, x_m^0) \in \Omega \subset \mathbb{R}^m$ is equal to the left-hyperderivative of $\widetilde{f}$ at any point $(x_0, x_1^0, x_2^0, \ldots, x_m^0) =: (x_0, x^0) \in \mathbb{R}^{m+1}$, and hence, is identically zero in the whole domain Ω:

$$\begin{aligned} {}'\widetilde{f}(x_0, x^0) &= \frac{1}{2}\overline{\mathcal{D}}_{CR}[\widetilde{f}\,](x_0, x^0) \\ &= \frac{\partial \widetilde{f}}{\partial x_0}(x_0, x^0) - \sum_{\ell=1}^{m} \mathbf{e}_\ell \frac{\partial \widetilde{f}}{\partial x_\ell}(x_0, x^0) \\ &= -\mathcal{D}_{\mathrm{Dir}}[f](x_1, x_2, \ldots, x_m) = 0 \,. \end{aligned}$$

That is, the set of Dirac-hyperholomorphic functions is a kind of the set of "Cauchy-Riemann-hyperholomorphic constants", and thus it is obviously not interesting to study the properties of the Cauchy-Riemann-hyperderivative of the Dirac-hyperholomorphic functions.

1.2.5. It is the aim of the paper to show that it is possible to develop another approach to the notion of hyperderivatives of a Dirac-hyperholomorphic function which is based on the same ideas but in such a way that the new hyperderivative does not vanish identically. We describe also the peculiarities of the situation and we explain why the Dirac-hyperholomorphic functions need to have several equivalent hyperderivatives.

2. The left-$\mathbf{e}_1$-Dirac-hyperderivative

2.1.

Since

$$\begin{aligned} \mathcal{D}_{\mathrm{Dir}} &= \sum_{\ell=1}^{m} \mathbf{e}_\ell \frac{\partial}{\partial x_\ell} \\ &= \mathbf{e}_1 \left(\frac{\partial}{\partial x_1} - \mathbf{e}_1\, \mathbf{e}_2 \frac{\partial}{\partial x_2} - \cdots - \mathbf{e}_1\, \mathbf{e}_m \frac{\partial}{\partial x_m} \right) \\ &= \mathbf{e}_1 \left(\frac{\partial}{\partial x_1} + \widehat{\mathbf{e}}_2 \frac{\partial}{\partial x_2} + \cdots + \widehat{\mathbf{e}}_m \frac{\partial}{\partial x_m} \right) \\ &= \mathbf{e}_1 \mathcal{D}_{CR}^1 \,, \end{aligned} \tag{2.1}$$

where

$$\mathcal{D}^1_{CR} = \frac{\partial}{\partial x_1} + \widehat{\mathbf{e}}_2 \frac{\partial}{\partial x_2} + \cdots + \widehat{\mathbf{e}}_m \frac{\partial}{\partial x_m}\,, \tag{2.2}$$

and $\widehat{\mathbf{e}}_2 := \mathbf{e}_2\,\mathbf{e}_1, \ldots, \widehat{\mathbf{e}}_m := \mathbf{e}_m\,\mathbf{e}_1$, then the sets of null-solutions of both operators coincide. At the moment, we can say that $\mathcal{D}^1_{CR}$ is a Cauchy-Riemann-type operator, which acts on $C^1(\Omega_m \subset \mathbb{R}^m; C\ell_{0,m})$. Observe that the restriction

$$\mathcal{D}^{1+}_{CR} := \mathcal{D}^1_{CR}|_{C^1(\Omega_m\subset\mathbb{R}^m;\, C\ell^+_{0,m})}\,,$$

is a "genuine" Cauchy-Riemann operator for the Clifford algebra $C\ell_{0,m-1} \cong C\ell^+_{0,m}$, whose imaginary units are $\widehat{\mathbf{e}}_2, \ldots, \widehat{\mathbf{e}}_m$, and $\widehat{\mathbf{e}}_0 := 1$.

2.2.

Thus, using the reasonings and the denotation given in Section 1.1 together with the decomposition of the Clifford algebra $C\ell_{0,m}$, given $f \in C^1(\Omega_m \subset \mathbb{R}^m; C\ell_{0,m})$, for any $x \in \Omega_m$ one has:

$$f(x) = f^+(x) + f^-(x)\,,$$

with $f^+(x) \in C\ell^+_{0,m}$ and $f^-(x) \in C\ell^-_{0,m}$. We know also that there exists $f_-(x) \in C\ell^+_{0,m}$ such that

$$f^-(x) = f_-(x)\,\alpha\,,$$

with α as in Section 1.1. Recall the notation $M^\mu : C^1(\Omega_m \subset \mathbb{R}^m;\ C\ell_{0,m-1}) \to C^1(\Omega_m \subset \mathbb{R}^m;\ C\ell_{0,m-1})$ for the operator that acts by $M^\mu[f](x) = M^\mu \circ f(x) := f(x)\,\mu$. Hence $f^+,\ M^\mu \circ f_- \in C^1(\Omega_m \subset \mathbb{R}^m;\ C\ell_{0,m-1})$ and

$$\begin{aligned} \mathcal{D}^1_{CR}[f](x) &= \mathcal{D}^1_{CR}[f^+](x) + \mathcal{D}^1_{CR}[M^\alpha \circ f_-](x) \\ &= \mathcal{D}^1_{CR}[f^+](x) + \mathcal{D}^1_{CR}[f_-](x)\,\alpha \\ &= \mathcal{D}^{1+}_{CR}[f^+](x) + \mathcal{D}^{1+}_{CR}[f_-](x)\,\alpha\,. \end{aligned} \tag{2.3}$$

2.3.

Adding the hypothesis that $f \in \ker \mathcal{D}_{\rm Dir} = \ker \mathcal{D}^1_{CR}$, i.e., $\mathcal{D}_{\rm Dir}[f](x) = 0$, from (2.3) and using the fact that $C\ell^+_{0,m} \cap C\ell^-_{0,m} = \{0\}$, one concludes that f^+ and f_- belong to $\ker \mathcal{D}^{1+}_{CR}$. It is clear that if $f^+,\ f_- \in \ker \mathcal{D}^{1+}_{CR}$ hence $f \in \ker \mathcal{D}_{\rm Dir}$. Summarizing we have

2.4. Proposition.

Let $f \in C^1(\Omega_m \subset \mathbb{R}^m;\ C\ell_{0,m})$ and take its decomposition $f = f^+ + f_-\,\alpha$, with f^+, f_- and α as before. Then $f \in \ker \mathcal{D}_{\rm Dir}$ if and only if $f^+,\ f_- \in \ker \mathcal{D}^{1+}_{CR}$.

2.5.

The analogs of (1.9) and (1.10) for the Clifford algebra $C\ell_{0,m-1} \cong C\ell^+_{0,m}$ with the imaginary units $\widehat{\mathbf{e}}_2, \dots, \widehat{\mathbf{e}}_m$ are:

$$\sigma_{x,1} = d\widehat{x}_1 - \widehat{\mathbf{e}}_2 d\widehat{x}_2 + \cdots + (-1)^{m-1}\widehat{\mathbf{e}}_m d\widehat{x}_m \,,$$

and

$$\tau_{x,1} = -\widehat{\mathbf{e}}_2 d\widehat{x}_{1,2} + \cdots + (-1)^{m-1}\widehat{\mathbf{e}}_m d\widehat{x}_{1,m} \,,$$

for which there holds for any $g \in C^1(\Omega_m \subset \mathbb{R}^m;\, C\ell_{0,\,m-1})$:

$$d(\tau_{x,1} g(x)) = \frac{1}{2}\left(\sigma_{x,1}\overline{\mathcal{D}}^{1+}_{CR}[\,g](x) - \overline{\sigma}_{x,1}\mathcal{D}^{1+}_{CR}[\,g](x)\right) . \tag{2.4}$$

In particular for $f = f^+ + M^\alpha \circ f_- \in C^1(\Omega_m \subset \mathbb{R}^m;\, C\ell_{0,m})$ given as before, one has:

$$\begin{aligned} d(\tau_{x,1}(f^+ + M^\alpha \circ f_-)(x)) &= d(\tau_{x,1} f^+(x)) \,+\, d(\tau_{x,1} f_-(x))\,\alpha \\ &= \frac{1}{2}\left(\sigma_{x,1}\overline{\mathcal{D}}^{1+}_{CR}[\,f^+](x) \,-\, \overline{\sigma}_{x,1}\mathcal{D}^{1+}_{CR}[\,f^+](x)\right) \\ &\qquad + \frac{1}{2}\left(\sigma_{x,1}\overline{\mathcal{D}}^{1+}_{CR}[\,f_-](x) - \overline{\sigma}_{x,1}\mathcal{D}^{1+}_{CR}[\,f_-](x)\right)\alpha \\ &= \frac{1}{2}\left(\sigma_{x,1}\left(\overline{\mathcal{D}}^{1+}_{CR}[\,f^+](x) \,+\, \overline{\mathcal{D}}^{1+}_{CR}[\,f_-](x)\,\alpha\right)\right. \\ &\qquad \left. -\ \overline{\sigma}_{x,1}\left(\mathcal{D}^{1+}_{CR}[\,f^+](x) + \mathcal{D}^{1+}_{CR}[\,f_-](x)\,\alpha\right)\right) \\ &= \frac{1}{2}\left(\sigma_{x,1}\overline{\mathcal{D}}^{1}_{CR}[\,f](x) - \overline{\sigma}_{x,1}\mathcal{D}^{1}_{CR}[\,f](x)\right) . \end{aligned} \tag{2.5}$$

Summarizing:

$$d(\tau_{x,1} f(x)) = \frac{1}{2}\left(\sigma_{x,1}\overline{\mathcal{D}}^{1}_{CR}[\,f](x) - \overline{\sigma}_{x,1}\mathcal{D}^{1}_{CR}[\,f](x)\right) . \tag{2.6}$$

So we are in a position to define a new hyperderivative for Dirac-hyperholomorphic functions.

2.6. Definition.

Let $f \in C^1(\Omega_m \subset \mathbb{R}^m;\, C\ell_{0,m})$. The function f is called left-$\mathbf{e}_1$-hyperderivable in Ω_m, if for any $x \in \Omega_m$ there is a Clifford number, denoted by $'f_{\mathbf{e}_1}(x)$, such that

$$d(\tau_{x,1} f(x)) = \sigma_{x,1}\ 'f_{\mathbf{e}_1}(x)\,. \tag{2.7}$$

The Clifford number $'f_{\mathbf{e}_1}(x)$ is named the left-$\mathbf{e}_1$-Dirac-hyperderivative of f at x.

Next theorem is immediate from (2.6).

2.7. Theorem.

Let $f \in C^1(\Omega_m \subset \mathbb{R}^m;\, C\ell_{0,m})$. The function f is Dirac-hyperholomorphic in Ω_m if and only if it is left-$\mathbf{e}_1$-Dirac-hyperderivable and for such a function

$$'f_{\mathbf{e}_1}(x) = \frac{1}{2}\,\overline{\mathcal{D}}^{1}_{CR}[f\,](x)\,, \quad \forall x \in \Omega_m\,. \tag{2.8}$$

2.8.

It should be noted that according to equation (2.5), Proposition 2.4 and Theorem 2.3.1 in [5], the left-Dirac-hyperderivative $'f_{\mathbf{e}_1}$ at any $x \in \Omega_m$ is determined by the hyperderivatives of the functions f^+, f_-, at the same point; and in fact the next theorem is valid.

2.9. Theorem.

Let $f \in C^1(\Omega_m \subset \mathbb{R}^m, C\ell_{0,m})$ and $f^+, f_- \in C^1(\Omega_m, C\ell_{0,m-1} \cong C\ell^+_{0,m})$, $\alpha \in C\ell_{0,m}$ such that, as before, $f = f^+ + M^\alpha \circ f_-$. Then f is left-$\mathbf{e}_1$-Dirac-hyperderivable in Ω_m if and only if f^+ and f_- are left-Cauchy-Riemann-hyperderivables in Ω_m, and moreover

$$'f_{\mathbf{e}_1}(x) = (f^+)'(x) + f'_-(x)\,\alpha\,, \quad \forall x \in \Omega_m\,, \tag{2.9}$$

where $(f^+)'$, f'_- are the left-Cauchy-Riemann-hyperderivatives of f^+, f_- respectively.

3. The left-$\mathbf{e}_1$-Dirac-hyperderivative as the limit of a quotient of increments

3.1.

Following the antecedents of this paper (see [7] and [5]), the left-$\mathbf{e}_1$-Dirac-hyperderivative can be seen as the limit of a quotient of the adequate increments of the function and of the variable. As it follows from [7], the key point here is that the increments have suitable dimensions. Let $x^0 \in \mathbb{R}^m$ and let $v_1, v_2, \dots, v_{m-1}$ be linearly independent vectors in $\mathbb{R}^m$. The $(m-1)$-dimensional parallelepiped Π with vertex x^0 and edges $v_1, v_2, \dots, v_{m-1}$ is defined by

$$\Pi := \left\{ x^0 + \sum_{\ell=1}^{m-1} t_\ell\, v_\ell \,|\, (t_1, t_2, \dots, t_{m-1}) \in [0,1]^{m-1} \right\},$$

and its boundary by

$$\partial\Pi := \left\{ x^0 + \sum_{\ell=1}^{m-1} t_\ell\, v_\ell \,|\, (t_1, t_2, \dots, t_{m-1}) \in \partial[0,1]^{m-1} \right\}.$$

3.2. Theorem.

Let $f \in \ker D_{\mathrm{Dir}}(\Omega_m)$ and $'f_{\mathbf{e}_1}(x)$ be its left-$\mathbf{e}_1$-Dirac-hyperderivative at a point $x \in \Omega_m \subset \mathbb{R}^m$. Then for any sequence $\{\Pi_k\}_{k=1}^\infty$ of oriented non-degenerated parallelepipeds with vertex x^0 and with $\lim\limits_{k\to\infty} \operatorname{diam} \Pi_k = 0$, there holds:

$$\lim_{k\to\infty} \left(\int_{\Pi_k} \sigma_{x,1} \right)^{-1} \left(\int_{\partial\Pi_k} \tau_{x,1}\, f(x) \right) = \,'f_{\mathbf{e}_1}(x^0)\,. \tag{3.1}$$

The proof follows from (2.6) and from Stokes' Theorem: first of all,

$$\lim_{k\to\infty}\left(\int_{\Pi_k}\sigma_{x,1}\right)^{-1}\left(\int_{\partial\Pi_k}\tau_{x,1}\,f(x)\right)=\lim_{k\to\infty}\left(\int_{\Pi_k}\sigma_{x,1}\right)^{-1}\cdot\int_{\Pi_k}d(\tau_{x,1}\,f(x))$$
$$=\lim_{k\to\infty}\left(\int_{\Pi_k}\sigma_{x,1}\right)^{-1}\cdot\int_{\Pi_k}\sigma_{x,1}\ {}'f_{\mathbf{e}_1}(x))\ ,$$

but also it is direct to prove that

$$'f_{\mathbf{e}_1}(x)=\ 'f_{\mathbf{e}_1}(x^0)+o(x-x^0)\,,$$

thus

$$\lim_{k\to\infty}\left(\int_{\Pi_k}\sigma_{x,1}\right)^{-1}\left(\int_{\partial\Pi_k}\tau_{x,1}\,f(x)\right)=\ 'f_{\mathbf{e}_1}(x^0)\,.$$

4. The directional left-$\mathbf{e}_1$-Dirac-hyperderivative

4.1. Definition

Let $L\subset\mathbb{R}^m$ be a hyperplane such that $L\cap\Omega_m\neq\emptyset$. The function $f:\Omega_m\subset\mathbb{R}^m\to C\ell_{0,m}$ is called left-$\mathbf{e}_1$-Dirac-hyperderivable at $x^0\in L\cap\Omega_m$ along the hyperplane L if for any sequence $\{\Pi_k\}_{k=1}^{\infty}$, with $\Pi_k\subset L$ and $\lim_{k\to\infty}\operatorname{diam}\Pi_k=0$, of parallelepipeds with vertex x^0, the limit

$$\lim_{k\to\infty}\left[\left(\int_{\Pi_k}\sigma_{x,1}\right)^{-1}\left(\int_{\partial\Pi_k}\tau_{x,1}\cdot f(x)\right)\right]\tag{4.1}$$

exists and is independent of the sequence $\{\Pi_k\}_{k=1}^{\infty}$. In this case the limit is denoted by $'f_{\mathbf{e}_1,\,L}(x^0)$ and is called the direcctional left-$\mathbf{e}_1$-Dirac-hyperderivative.

4.2.

Note that the limits in (4.1) and (3.1) are quite similar. The crucial difference between them is the fact that the parallelepipeds considered in (4.1) are "caught" in the hyperplane L.

4.3.

Given as before an $(m-1)$-dimensional hyperplane $L\subset\mathbb{R}^m$, let

$$\gamma(x):=\sum_{\ell=1}^{m}n_\ell x_\ell+d=0$$

be its equation, with $\widehat{n} = (n_1, n_2, \ldots, n_m)$ the unitary normal vector, and $d \in \mathbb{R}$. Applying formula (2.6) to the function γ, there holds:

$$d(\tau_{x,1}\,\gamma(x)) = \frac{1}{2}\left(\sigma_{x,1}\,\overline{\mathcal{D}}^{\,1}_{CR}[\gamma](x) - \overline{\sigma}_{x,1}\,\mathcal{D}^{1}_{CR}[\gamma](x)\right) = \frac{1}{2}\left(\sigma_{x,1}\overline{\breve{n}} - \overline{\sigma}_{x,1}\,\breve{n}\right),$$

where $\breve{n} := n_1 + n_2\,\widehat{\mathbf{e}}_2 + \cdots + n_3\,\widehat{\mathbf{e}}_m$. This differential form is identically zero on L, thus

$$\sigma_{x,1}\,\overline{\breve{n}} = \overline{\sigma}_{x,1}\,\breve{n}\,, \quad \text{for} \quad x \in L\,.$$

4.4.

Let us combine the latter fact with Stokes' Theorem. Consider a function f which satisfies the conditions in Definition 4.1, then

$$\left(\int_{\Pi_k}\sigma_{x,1}\right)^{-1}\left(\int_{\partial\Pi_k}\tau_{x,1}\,f(x)\right) = \frac{1}{2}\left(\int_{\Pi_k}\sigma_{x,1}\right)^{-1}\int_{\Pi_k}\sigma_{x,1}\,\left(\overline{\mathcal{D}}^{\,1}_{CR}[f](x) - (\overline{\breve{n}})^2\,\mathcal{D}^{1}_{CR}[f](x)\right).$$

This formula implies two facts which one would expect from a suitable notion of directional derivative and which we present below, after this comment. The first fact is related with functions of class $C^1(\Omega_m \subset \mathbb{R}^m; C\ell_{0,m})$ and claims that these functions possess the left-$\mathbf{e}_1$-Dirac-hyperderivative along any $(m-1)$-dimensional hyperplane that intersects the domain, and it is given in terms of the direction of the corresponding hyperplane. The second fact says that if the function $f : \Omega_m \subset \mathbb{R}^m \to C\ell_{0,m}$ is Dirac-hyperholomorphic then the directional left-$\mathbf{e}_1$-Dirac-hyperderivative does not depend on the direction of the hyperplane.

4.5. Theorem.

Let $f \in C^1(\Omega_m \subset \mathbb{R}^m; C\ell_{0,m})$. Then f is left-$\mathbf{e}_1$-Dirac-hyperderivable along any $(m-1)$-dimensional hyperplane L at every $x^0 \in L \cap \Omega_m$, and the left-$\mathbf{e}_1$-Dirac-hyperderivative along the plane L is given by

$$'f_{\mathbf{e}_1,\,L}(x^0) = \frac{1}{2}\left(\overline{\mathcal{D}}^{\,1}_{CR}[f](x^0) + \left(\overline{\breve{n}}\right)^2\,D^{1}_{CR}[f](x^0)\right). \tag{4.2}$$

4.6. Corollary.

Let $f \in C^1(\Omega_m \subset \mathbb{R}^m; C\ell_{0,m})$. Then f is Dirac-hyperholomorphic in Ω_m if and only if $\forall\, x^0 \in \Omega_m$ the directional hyperderivative $'f_{\mathbf{e}_1,L}(x^0)$ is independent of the hyperplane L, with $x^0 \in L \cap \Omega_m$. Moreover, in this case all the directional hyperderivatives $'f_{\mathbf{e}_1,L}(x^0)$ are equal to $'f_{\mathbf{e}_1}(x^0)$.

5. The left-$\mathbf{e}_1$-Dirac-hyperderivative and the Clifford-Cauchy-type integral

5.1.

As in the previous papers, an application of the hyperderivative to the Clifford-Cauchy-type integral proves to be instructive. In order to compute the left-$\mathbf{e}_1$-Dirac-hyperderivative of the Clifford-Cauchy-type integral, it is neccessary to establish the corresponding relations between the Clifford-Cauchy kernel and the right- and left-Cauchy-Riemann operators.

5.2. The right-hand side operators

The right-hand side Cauchy-Riemann operator, which acts on functions $C^1(\Lambda \subset \mathbb{R}^{m+1};\ C\ell_{0,m})$ is given by

$$\mathcal{D}_r := \sum_{\ell=0}^{m} M^{\mathbf{e}_\ell} \frac{\partial}{\partial x_\ell}\,.$$

Analogously the right-hand side Dirac operator is given as

$$\mathcal{D}_{\mathrm{Dir},r} := \sum_{\ell=1}^{m} M^{\mathbf{e}_\ell} \frac{\partial}{\partial x_\ell}\,.$$

It acts on $C^1(\Omega_m \subset \mathbb{R}^m;\ C\ell_{0,m})$, and it can be written as

$$\mathcal{D}_{\mathrm{Dir},r} = M^{\mathbf{e}_1}\left(\frac{\partial}{\partial x_1} - \sum_{\ell=2}^{m} M^{\widehat{\mathbf{e}}_\ell} \frac{\partial}{\partial x_\ell}\right).$$

Define

$$\mathcal{D}^1_{CR,r} := \frac{\partial}{\partial x_1} - \sum_{\ell=2}^{m} M^{\widehat{\mathbf{e}}_\ell} \frac{\partial}{\partial x_\ell}\,,$$

and its conjugate:

$$\overline{\mathcal{D}}^1_{CR,r} := \frac{\partial}{\partial x_1} + \sum_{\ell=2}^{m} M^{\widehat{\mathbf{e}}_\ell} \frac{\partial}{\partial x_\ell}\,,$$

thus, similarly to what happens with the left-hand side operators, the sets of the null-solutions of $\mathcal{D}_{\mathrm{Dir},r}$ and $\mathcal{D}^1_{CR,r}$ coincide. Moreover, the right-hand side analog of (2.6) is valid:

$$d(f(x)\,\tau_{x,1}) = \frac{1}{2}\left(\,\mathcal{D}^1_{CR,r}[f](x)\,\sigma_{x,1} \,-\, \overline{\mathcal{D}}^1_{CR,r}[f](x)\,\overline{\sigma}_{x,1}\right). \tag{5.1}$$

The immediate right-hand side analogs of Definition 2.6 and Theorem 2.7 follow:

5.3. Definition.

Let $f \in C^1(\Omega_m \subset \mathbb{R}^m;\ C\ell_{0,n})$. The function f is called right-$\mathbf{e}_1$-Dirac-hyperderivable in Ω_m, if for any $x \in \Omega_m$ there is a Clifford number denoted by $f'_{\mathbf{e}_1}(x)$ such that

$$d(f(x)\,\tau_{x,\,1}) = f'_{\mathbf{e}_1}(x)\,\overline{\sigma}_{x,\,1}\,.$$

5.4. Theorem.

Let $f \in C^1(\Omega_m \subset \mathbb{R}^m; C\ell_{0,m})$. The function f is right-Dirac-hyperholomorphic in Ω_m if and only if it is right-$\mathbf{e}_1$-Dirac-hyperderivable in Ω_m and in this case for any $x \in \Omega_m$ it follows:

$$f'_{\mathbf{e}_1}(x) = -\frac{1}{2}\overline{\mathcal{D}}^1_{CR,r}[f\,](x)\,.$$

5.5.

From (2.6) and (5.1) there holds:

$$\begin{aligned} d(f(x)\,\tau_{x,1}\,g(x)) = \frac{1}{2}(\mathcal{D}^1_{CR,r}[f\,](x)\,\sigma_{x,\,1}\,g(x) \;-\; \overline{\mathcal{D}}^1_{CR,r}[f\,](x)\,\overline{\sigma}_{x,\,1}\,g(x) \\ +(-1)^{m-2}f(x)\,\sigma_{x,\,1}\,\overline{\mathcal{D}}^1_{CR}[g\,](x) \\ +\,(-1)^{m-1}f(x)\,\overline{\sigma}_{x,\,1}\,\mathcal{D}^1_{CR}[g](x))\,. \end{aligned} \tag{5.2}$$

5.6.

Let us recall that

$$E(y-x) = \frac{\overline{y-x}}{A_m\ \mid y-x\mid^m}\,,$$

with A_m the surface area of the unit sphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$, is the Cauchy kernel, which is left- and right-Dirac-hyperholomorphic in $\mathbb{R}^m \setminus \{x\}$. Hence from (5.2) one has:

$$\begin{aligned} d_y(E(y-x)\,\tau_{y,\,1}\,f(y)) = \frac{1}{2}\Big(\;-\;\overline{\mathcal{D}}^1_{CR,r}\,[E](y-x)\,\overline{\sigma}_{y,\,1}\,f(y) \\ +\,(-1)^{m-2}E(y-x)\,\sigma_{y,\,1}\,\overline{\mathcal{D}}^1_{CR}[f](y) \\ +\,(-1)^{m-1}E(y-x)\,\overline{\sigma}_{y,\,1}\,\mathcal{D}^1_{CR}[f](y)\Big)\,. \end{aligned} \tag{5.3}$$

There are some relations between the Cauchy kernel and the operators $\overline{\mathcal{D}}^1_{CR}$ and $\overline{\mathcal{D}}^1_{CR,r}$ that we shall use:

$$\overline{\mathcal{D}}^1_{CR,r,y}[E(y-x)] = -\overline{\mathcal{D}}^1_{CR,r,x}[E(y-x)] = -\overline{\mathcal{D}}^1_{CR,x}[E(y-x)]\,. \tag{5.4}$$

5.7.

Let $\Omega_m \subset \mathbb{R}^m$ and let $\Gamma := \{\, y \in \mathbb{R}^m \mid \varrho(y) = 0\,\}$ be its smooth boundary. Integrating (5.3) over Γ, on the left side we get:

$$\int_\Gamma d_y\,(E(y-x)\,\tau_{y,1}\,f(y)) = 0\,.$$

Hence

$$\begin{aligned} \int_\Gamma \overline{\mathcal{D}}^1_{CR,r,y}\,[E](y-x)\,\overline{\sigma}_{y,\,1}\,f(y) = (-1)^{m-2}\int_\Gamma E(y-x)\,\sigma_{y,\,1}\,\overline{\mathcal{D}}^1_{CR,y}[f](y) \\ +\,(-1)^{m-1}\int_\Gamma E(y-x)\,\overline{\sigma}_{y,\,1}\,\mathcal{D}^1_{CR,y}[f](y)\,; \end{aligned}$$

using (5.4) one has:

$$-\overline{\mathcal{D}}^1_{CR,x}\int_\Gamma E(y-x)\,\overline{\sigma}_{y,1}\,f(y) = (-1)^{m-2}\int_\Gamma E(y-x)\,\sigma_{y,1}\,\overline{\mathcal{D}}^1_{CR,y}[f](y) \qquad (5.5)$$
$$+\ (-1)^{m-1}\int_\Gamma E(y-x)\,\overline{\sigma}_{y,1}\,\mathcal{D}^1_{CR,y}[f](y)\,.$$

Observe that the function ϱ is identically zero on Γ. Combining this fact and formula (2.6) applied to ϱ, one concludes that

$$\overline{\sigma}_{y,1} = \sigma_{y,1}\,\overline{\mathcal{D}}^1_{CR,y}[\varrho(y)]\,\left(\mathcal{D}^1_{CR,y}[\varrho(y)]\right)^{-1} =: \sigma_{y,1}\,V^1_{\Gamma(y)}\,,$$

where we have defined $V^1_{\Gamma(y)} := \overline{\mathcal{D}}^1_{CR,y}[\varrho(y)]\,\left(\mathcal{D}^1_{CR,y}[\varrho(y)]\right)^{-1}$. Thus (5.5) becomes:

$$\overline{\mathcal{D}}^1_{CR,x}\int_\Gamma E(y-x)\,\overline{\sigma}_{y,1}\,f(y) \qquad (5.6)$$
$$= \int_\Gamma E(y-x)\,\sigma_{y,1}\,\left((-1)^{m-1}\overline{\mathcal{D}}^1_{CR,y}[f(y)]\ +\ (-1)^m V^1_{\Gamma(y)}\,\mathcal{D}^1_{CR,y}[f(y)]\right)\,.$$

Hence next theorem has been proved.

5.8. Theorem.
Let $\Omega_m \subset \mathbb{R}^m$ be a simply connected domain with boundary $\Gamma := \{\,y \in \mathbb{R}^m \mid \varrho(y) = 0\,\}$, where $\varrho \in C^1(\mathbb{R}^m, \mathbb{R})$, grad$(\varrho)|\Gamma(y) \neq 0$ for all $y \in \Gamma$ and let $f \in C^1(\Gamma, C\ell_{0,m})$. Then for all $x \notin \Gamma$ the equallity (5.6) holds, that is, the $\mathbf{e}_1$-Dirac–hyperderivative of the Cauchy-type integral is also a Cauchy-type integral but now with the “derived density”.

An immediate consequence is

5.9. Corollary.
Let $p \in \mathbb{N}$, $f \in C^p(\Gamma, C\ell_{0,m})$ and $\varrho \in C^p(\mathbb{R}^m, \mathbb{R})$. Then for all $x \notin \Gamma$ there follows:

$$\left(\overline{\mathcal{D}}^1_{CR,x}\right)^p \int_\Gamma E(y-x)\,\overline{\sigma}_{y,1}\,f(y)$$
$$= \left(\int_\Gamma E(y-x)\,\overline{\sigma}_{y,1}\,f(y)\right)^{(p)}$$
$$= \int_\Gamma E(y-x)\,\sigma_{y,1}\,\left(\overline{\mathcal{D}}^1_{CR,y}\ -\ V^1_{\Gamma(y)}\,\mathcal{D}^1_{CR,y}\right)^p [f(y)]\,.$$

6. The left $\mathbf{e}_\ell$-hyperderivatives

In the previous sections the reasoning was concentrated around the imaginary unit $\mathbf{e}_1$. Let us show briefly that $\mathbf{e}_k$ with $k \in \{2, 3, \dots, m\}$ may play a similar role.

For each $k \in \{2, 3, \dots, m\}$, the Dirac operator may be written as follows:

$$
\begin{aligned}
\mathcal{D}_{\mathrm{Dir}} &= \sum_{\ell=1}^{m} \mathbf{e}_\ell \frac{\partial}{\partial x_\ell} \\
&= \mathbf{e}_k \left(-\mathbf{e}_k \, \mathbf{e}_1 \frac{\partial}{\partial x_1} - \mathbf{e}_k \, \mathbf{e}_2 \frac{\partial}{\partial x_2} - \cdots - \mathbf{e}_k \, \mathbf{e}_{k-1} \frac{\partial}{\partial x_{k-1}} \right. \\
&\qquad \left. + \frac{\partial}{\partial x_k} - \mathbf{e}_k \, \mathbf{e}_{k+1} \frac{\partial}{\partial x_{k+1}} - \cdots - \mathbf{e}_k \, \mathbf{e}_m \frac{\partial}{\partial x_m} \right) \\
&= \mathbf{e}_k \left(\mathbf{e}_{1,k} \frac{\partial}{\partial x_1} + \mathbf{e}_{2,k} \frac{\partial}{\partial x_2} + \cdots + \mathbf{e}_{k-1,k} \frac{\partial}{\partial x_{k-1}} \right. \\
&\qquad \left. + \frac{\partial}{\partial x_k} + \mathbf{e}_{k+1,k} \frac{\partial}{\partial x_{k+1}} + \cdots + \mathbf{e}_{m,k} \frac{\partial}{\partial x_m} \right) \\
&= \mathbf{e}_k \, \mathcal{D}_{CR}^k \, , \qquad (6.1)
\end{aligned}
$$

where each $\mathcal{D}_{CR}^k$ is the Cauchy-Riemann-type operator:

$$
\begin{aligned}
\mathcal{D}_{CR}^k = \mathbf{e}_{1,k} \frac{\partial}{\partial x_1} + \mathbf{e}_{2,k} \frac{\partial}{\partial x_2} + \cdots + \mathbf{e}_{k-1,k} \frac{\partial}{\partial x_{k-1}} \\
+ \frac{\partial}{\partial x_k} + \mathbf{e}_{k+1,k} \frac{\partial}{\partial x_{k+1}} + \cdots + \mathbf{e}_{m,k} \frac{\partial}{\partial x_m} \, .
\end{aligned}
$$

These operators keep characterizing the Dirac-hyperholomorphic functions:

$$
\ker \mathcal{D}_{\mathrm{Dir}} = \ker \mathcal{D}_{CR}^k \subset C^1(\Omega_m \subset \mathbb{R}^m \, ; \, C\ell_{0,m} \,).
$$

Considering the $(m-1)$-diferential form of hypersurface:

$$
(\,(-1)^m d\widehat{x}_1, \, (-1)^{m-1} d\widehat{x}_2, \dots, d\widehat{x}_{m-1}, -d\widehat{x}_m) \, , \qquad (6.2)
$$

where $d\widehat{x}_\ell$ were defined in Subsection 1.2.3, one “Cliffordize” it as usual for every value of k, obtaining the $(m-1)$-differential form:

$$
\begin{aligned}
\sigma_{x,\,k} := (-1)^m \mathbf{e}_{1,k} d\widehat{x}_1 + (-1)^{m-1} \mathbf{e}_{2,k} d\widehat{x}_2 + \cdots + (-1)^{m-k+2} \mathbf{e}_{k-1,k} d\widehat{x}_{k-1} \\
+ (-1)^{m-k+1} d\widehat{x}_k + (-1)^{m-k} \mathbf{e}_{k+1,k} d\widehat{x}_{k+1} + \cdots \\
\cdots + \, \mathbf{e}_{m-1} d\widehat{x}_{m-1} - \mathbf{e}_{m,k} d\widehat{x}_m \, .
\end{aligned}
$$

As usual also, we get from the latter the $(m-2)$-differential form:

$$
\begin{aligned}
\tau_{x,\,k} := (-1)^{m-1} \mathbf{e}_{1,k} d\widehat{x}_{1,k} + (-1)^{m-2} \mathbf{e}_{2,k} d\widehat{x}_{2,k} + \cdots \\
\cdots + (-1)^{m-k+1} \mathbf{e}_{k-1,k} d\widehat{x}_{k-1,k} \; + \; (-1)^{m-k-1} \mathbf{e}_{k+1,k} d\widehat{x}_{k+1,k} + \cdots \\
\cdots - \; \mathbf{e}_{m-1} d\widehat{x}_{m-1,k} + \mathbf{e}_{m,k} d\widehat{x}_{m,k} \, .
\end{aligned}
$$

Again, $d\widehat{x}_{\ell,k}$ were defined in Subsection 1.2.3. The following crucial correlations are valid for any $f \in C^1(\Omega_m \subset \mathbb{R}^m \, ; \, C\ell_{0,m} \,)$:

$$
d(\tau_{x,\,k} \, f(x)) = \frac{1}{2} \left(\sigma_{x,\,k} \, \overline{\mathcal{D}}_{CR}^k[f](x) \; - \; \overline{\sigma}_{x,\,k} \, \mathcal{D}_{CR}^k[f](x) \right) .
$$

Thus the left $\mathbf{e}_k$-Dirac-hyperderivatives are defined in an exact analogy to the previous sections, concluding that:

$$'f_{\mathbf{e}_k}(x) = -\frac{1}{2}\overline{\mathcal{D}}^k_{CR}[f](x)\,,$$

for any $x \in \Omega$.

The rest of definitions and theorems related with the $\mathbf{e}_k$-hyperderivatives can be given also.

7. Comparison with one complex variable case

The existence of several hyperderivatives for the same Dirac-hyperholomorphic function may cause an impression that we have a phenomenon with no analogues in the classical one complex variable theory. Let us show briefly that this is not the case, i.e., the phenomenon does have its antecedents.

Consider a function $f = u + \mathbf{i}\,v : \Omega \subset \mathbb{R}^2 \to \mathbb{C}$, that is, the function f takes complex values but depends on two real variables, not on a complex one, that is, its domain is not endowed with any complex numbers structure; this mimics the previously considered Cliffordian situation. For such functions of class C^1 the Cauchy-Riemann operator is well defined and may be used in order to define the class $Hol(\Omega)$ of "holomorphic" in Ω functions:

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + \mathbf{i}\,\frac{\partial}{\partial y}\right)[f] \,=\, 0\,. \tag{7.1}$$

The same class is determined by another operator:

$$\frac{1}{2}\left(\frac{\partial}{\partial y} - \mathbf{i}\,\frac{\partial}{\partial x}\right)[f] \,=\, 0\,, \tag{7.2}$$

compare with $\mathcal{D}^1_{CR}$. But since we assume no complex numbers structure in Ω the derivative as the limit of a special quotient cannot be introduced for $f \in Hol(\Omega)$. The equation (7.1) corresponds to the complex variable $z := x + \mathbf{i}\,y$ which generates the function $f_1 : z \in \Omega_1 := \{x + \mathbf{i}\,y \mid (x, y) \in \Omega\} \mapsto f(x, y) \in \mathbb{C}$, and equation (7.2) corresponds to the complex variable $\zeta := y - \mathbf{i}\,x$ which generates the function $f_2 : \zeta \in \Omega_2 := \{\zeta = y - \mathbf{i}\,x \mid (x, y) \in \Omega\} \mapsto f(x, y) \in \mathbb{C}$. Thus, to any $f \in Hol(\Omega)$ we associate <u>two</u> complex functions (that is, both go from $\mathbb{C}$ to $\mathbb{C}$) each of them having a derivative in the usual sense: for $z \in \Omega_1$ and $\zeta_0 = -\mathbf{i}\,z_0 \in \Omega_2$ there exist the complex numbers $f_1'(z_0)$ and $f_2'(\zeta_0)$ but they are different in general; it can be shown that $f_2'(\zeta_0) = \mathbf{i}\,f_1'(z_0)$. Both $f_1'(z_0)$ and $f_2'(\zeta_0)$ can be equally called "the derivative of f at $(x_0,\, y_0)$", hence we are in exactly the same situation as for Dirac-hyperholomorphic functions.

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On the Discrete Cosine Transform of Weakly Stationary Signals

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Abstract. The Discrete Cosine Transform (DCT) is used in a large variety of applications, due to its near-optimal properties for representing and decorrelating random signals. This paper describes a useful but relatively unknown property of this transform, when applied to weakly stationary signals. The transformed autocorrelation matrix has half of its elements equal to zero. This means that it is possible to improve current DCT signal processing systems by means of more efficient implementation and algorithms.

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1. Introduction

The Discrete Cosine Transform (DCT) [1] was first described in 1974. It is derived from approximations to the eigenvectors of autocorrelation matrices for Markov-1 signals [2], and furthermore, its equivalence to Markov random processes has been well established [3].

The DCT is traditionally regarded as a powerful signal decorrelator, mainly due to its *energy packing efficiency* and *decorrelation* properties, among others. The DCT approximately diagonalizes the autocorrelation matrix of the input signal [4, 5], and is known to be asymptotically equivalent to the KLT [6]. These properties and the availability of fast computation algorithms, have turned it into the current *de facto* standard mapping for image and video compression.

This paper shows that the DCT of a symmetric Toeplitz matrix, has roughly half of its elements equal to zero as outlined in [7]. This means that the DCT achieves perfect decorrelation for some samples of weakly stationary signals. Possible applications of this property include performance improvement for fast DCT algorithms [2, 8, 9], especially when involving autocorrelation matrices [7], efficient computation using relationships to other transforms [10], as well as speech coding [11], analysis and recognition systems [12].

This paper is organized as follows. Section II defines basic concepts about the statistics of stationary random processes. In Section III we define the DCT, and describe the general nomenclature. Section IV establishes that half of the elements of a transformed symmetric Toeplitz matrix are zero.

2. Stationary random processes

Let the sequence $x(n)$ represent a discrete random signal, such that its value for any choice of the integer parameter n is a random variable. The underlying model that represents the random sequence is known as a *random process* or *stochastic process.*

Now, suppose that a random variable $x(n)$ has a probability function f_x, and let x_n be the sum of all instances belonging to $x(n)$ at each value of n. Then, the result is a deterministic sequence [13] which represents the mean of the associated random process.

Definition 2.1. Formally, the mean of a random variable is defined as

$$m(n) = E\{x(n)\} = \int_{-\infty}^{\infty} x_n f_x(x(n)) dx_n \tag{2.1}$$

where, for any fixed value of n, $f_x(x(n))$ represents the density function of the random variable $x(n)$, and $E\{\cdot\}$ represents the expected value operator. This average will be referred to as *ensemble average.*

Now, assume each sequence $x(n)$, to exist for all n from $-\infty$ to ∞, and let $\langle x(n)\rangle_{n_0,n_1}$ denote the average of the sequence $x(n)$ computed from $n = n_0$ to $n = n_1$.

Definition 2.2. Define the *signal average* as

$$\langle x(n)\rangle = \lim_{M\to\infty} \langle x(n)\rangle_{-M,M} = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x(n) \; . \tag{2.2}$$

A random process is said to be *stationary* (in the *strict* sense) if its statistical description is not a function of n. In particular, if the mean and the variance are independent of n, the process is said to be *wide sense* stationary or *weakly* stationary.

In addition, a random process is said to be *ergodic*, if its moments computed as signal averages are equal, with probability 1, to the corresponding ensemble averages. A random process that satisfies the condition

$$\langle x(n)\rangle \;=\; E\{x(n)\} \tag{2.3}$$

is said to be *ergodic in the mean*, whereas one that satisfies

$$\langle x(n)x(n+l)\rangle \;=\; E\{x(n)x(n+l)\} \tag{2.4}$$

is said to be *ergodic in correlation.* The concept or ergodicity is often used in applications where the experiments cannot be repeated, making it necessary to infer the statistics of a particular process from a single data sequence.

In a similar manner, we can define the variance as the expected value of $|x(n) - m(n)|^2$.

2.1. First and second moments of stationary signals

The correlation between any two samples of a random process $x(n_1)$ and $x(n_0)$ is expressed by the *correlation function* (also called the *auto*correlation function)

$$\mathrm{R}_x(n_1, n_0) = E\{x(n_1)x(n_0)\} \ . \tag{2.5}$$

In a similar way, the covariance between any two samples of a random process is expressed by the *covariance function* (also called the *auto*covariance function)

$$\mathrm{C}_x(n_1, n_0) = E\{[x(n_1) - m_x(n_1)][x(n_0) - m_x(n_0)]\} \ , \tag{2.6}$$

which is related to the autocorrelation by the expression

$$\mathrm{R}_x(n_1, n_0) = \mathrm{C}_x(n_1, n_0) + m_x(n_1)m_x(n_0) \ . \tag{2.7}$$

Since for a stationary random process the probability density is a function of the spacing between samples only, it must be true that

$$m_x(n) = m_x \text{ (a constant)} \tag{2.8}$$

and

$$\mathrm{R}_x(n_1, n_0) = R_x(n_1 - n_0) = R_x(l) \ . \tag{2.9}$$

where R_x is a new function[1] that depends on only the difference $l = n_1 - n_0$, sometimes called the *lag*. This implies that for a stationary random process, the autocovariance is a function of only the lag

$$\mathrm{C}_x(n_1, n_0) = C_x(n_1 - n_0) = C_x(l) \ . \tag{2.10}$$

Definition 2.3. A random process $x(n)$ is said to be *wide-sense stationary* or *weakly stationary* if the mean is a constant m_x and the correlation function is a function only of the spacing between the samples, $\mathrm{R}_x(n_1, n_0) = R_x(n_1 - n_0)$.

2.2. Correlation and covariance matrices

Let $\mathbf{x}$ be a data vector consisting of N samples of the random process x,

$$\mathbf{x}^T = \begin{bmatrix} x(0) \ x(1) \ \cdots \ x(N-1) \end{bmatrix} \ . \tag{2.11}$$

The *mean vector* is given by

$$\mathbf{m}_x = E\{x\} = \begin{bmatrix} m_x(0) \\ m_x(1) \\ \vdots \\ m_x(N-1) \end{bmatrix} \ . \tag{2.12}$$

[1]A math italic font is used for the function that depends on the difference $n_1 - n_0$ while a roman font is used for the original function $\mathrm{R}_x(n_1, n_0)$.

For a stationary random process the mean vector has all of its components equal to the same constant m_x.

The *correlation matrix* of the random process is defined by

$$\begin{aligned}\mathbf{R}_x &= E\{\mathbf{x}\mathbf{x}^T\}\\ &= \begin{bmatrix} E\{x^2(0)\} & E\{x(0)x(1)\} & \cdots & E\{x(0)x(N-1)\} \\ E\{x(1)x(0)\} & E\{x^2(1)\} & \cdots & E\{x(1)x(N-1)\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{x(N-1)x(0)\} & E\{x(N-1)x(1)\} & \cdots & E\{x^2(N-1)\} \end{bmatrix}\\ &= \begin{bmatrix} \mathrm{R}_x(0,0) & \mathrm{R}_x(0,1) & \cdots & \mathrm{R}_x(0,N-1) \\ \mathrm{R}_x(1,0) & \mathrm{R}_x(1,1) & \cdots & \mathrm{R}_x(1,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{R}_x(N-1,0) & \mathrm{R}_x(N-1,1) & \cdots & \mathrm{R}_x(N-1,N-1) \end{bmatrix}\end{aligned} \tag{2.13}$$

This matrix is completely specified by the correlation function of the random process. For a stationary random process the correlation matrix depends only on a single argument, as stated in eq. (2.9). Hence, it takes the special form [13]

$$\mathbf{R}_x = \begin{bmatrix} \rho_0 & \rho_1 & \rho_2 & \rho_3 & \\ \rho_1 & \rho_0 & \rho_1 & \rho_2 & \ddots \\ \rho_2 & \rho_1 & \rho_0 & \rho_1 & \ddots \\ \rho_3 & \rho_2 & \rho_1 & \rho_0 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{2.14}$$

where $\rho_l = R_x(l) = E\{x(n_0)x(n_0+l)\}$ denotes the autocorrelation for lag l, for arbitrary n_0 (since it depends only on l). Note that all elements on each of the main diagonals of this matrix are equal. This form is known as a *Toeplitz* matrix. All the elements on a given diagonal represent correlations between terms of the random process with the same lag separation. Thus, the correlation matrix for any stationary random process is always a symmetric Toeplitz matrix.

Note that the *covariance matrix* defined by

$$\mathbf{C} = E\{(x - \mathbf{m}_x)(x - \mathbf{m}_x)^T\} \tag{2.15}$$

also takes the symmetric Toeplitz form. Throughout this paper we assume without loss of generality, that the random process x has zero mean and unity variance. This implies that the autocovariance and the autocorrelation matrices of our signals are identical, and both have unity diagonal, namely $\rho_0 = 1$.

3. The discrete cosine transform

The discrete cosine transform (DCT) maps a finite *time domain* sequence $x(n)$ into another *frequency domain* finite sequence $z(k)$. There are four standard types of DCT. The most widely used is the DCT-II, defined in [2] as

$$z(k) = c_l \sum_{n=0}^{N-1} x(n) \cos \frac{\pi(2n+1)k}{2N} \ , \tag{3.1}$$

where $c_l = \sqrt{\frac{2}{N}} - \delta(l)\frac{\sqrt{2}-1}{\sqrt{N}}$, for $l = 0, 1, \ldots, N-1$, and $\delta(l)$ is the discrete delta function.

The DCT-II can also be expressed in matrix form. It transforms a real vector $\mathbf{x}$ into another real vector $\mathbf{z}$, using the relation

$$\mathbf{z} = \mathbf{DWx} \ , \tag{3.2}$$

with

$$\mathbf{x} = \begin{bmatrix} x(0) \ x(1) \ \cdots \ x(N-1) \end{bmatrix}^T \ , \tag{3.3a}$$

$$\mathbf{z} = \begin{bmatrix} z(0) \ z(1) \ \cdots \ z(N-1) \end{bmatrix}^T \ , \tag{3.3b}$$

the symbol ^{T} denotes transposition,

$$\begin{aligned} \mathbf{D} &= \operatorname{diag}(c_0, c_1, \ \ldots \ , c_{N-1}) \\ &= \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \sqrt{2}\, \mathbf{I}_{N-1} \end{bmatrix} \ , \end{aligned} \tag{3.4}$$

the *DCT matrix*

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_0 \\ \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_{N-1} \end{bmatrix} \ , \tag{3.5}$$

where $\mathbf{0}$ represents a vector of zeros, $\mathbf{I}_{N-1}$ is an identity matrix of size $N-1$,

$$\mathbf{w}_j = \begin{bmatrix} u_0 \ u_1 \ \cdots \ u_{N-1} \end{bmatrix} \ , \tag{3.6}$$

for

$$u_q = \cos \frac{(2q+1)}{2N} j\pi \ , \tag{3.7}$$

and $j, q = 0, \ldots, N-1$.

Thus, the autocorrelation matrix of the transformed sequence $\mathbf{z} = \mathbf{DWx}$ is given by

$$\begin{aligned} \mathbf{R}_z &= E\{\mathbf{z}\mathbf{z}^T\} \\ &= E\{\mathbf{DWx}(\mathbf{DWx})^T\} = \mathbf{DWR}_x\mathbf{W}^T\mathbf{D}^T \ . \end{aligned} \tag{3.8}$$

Note that the constant matrix $\mathbf{D}$ represents scaling factors, which can be dropped without loss of generality.

Now, let $r_z(j,k)$ denote the elements of $\mathbf{R}_z$, for $j, k = 0, \ldots, N-1$. It can be easily shown that

$$r_z(j,k) = \mathbf{w}_j \mathbf{R}_x \mathbf{w}_k^T \ . \tag{3.9}$$

In the next section we show that

$$r_z(j,k) = 0 \ , \tag{3.10}$$

for symmetric Toeplitz $\mathbf{R}_x$, when $j+k$ is odd.

4. Zeros on the diagonals

In this section we show that half of the elements of a DCT-II-transformed symmetric Toeplitz matrix, are zero.

4.1. Nomenclature and fundamental identities

Let

$$\mathbf{w}_k = \begin{bmatrix} v_0 & v_1 & \cdots & v_{N-1} \end{bmatrix} \tag{4.1}$$

where

$$v_q = \cos \frac{(2q+1)}{2N} k\pi \ , \tag{4.2}$$

for $k, q = 0, \ldots, N-1$.

Substituting (3.7) and (4.1) in (3.9) can find that each element $r_z(j,k)$ of the transformed matrix $\mathbf{R}_z$ can be expressed as:

$$r_z(j,k) = \sum_{q=0}^{N-1} u_q v_q + \sum_{p=1}^{N-1} \rho_p \sum_{q=0}^{N-p-1} u_q v_{q+p} + u_{q+p} v_q \ . \tag{4.3}$$

Taking advantage of well-known properties of the cosine function, one can easily show that

$$\cos \frac{2(N-q-1)+1}{2N} j\pi = (-1)^j \cos \frac{2q+1}{2N} j\pi \ , \tag{4.4}$$

which implies that

$$u_{N-q-1} = (-1)^j u_q \ , \tag{4.5}$$

and in the same fashion,

$$v_{N-q-1} = (-1)^k v_q \ . \tag{4.6}$$

Similar manipulations lead to the following identities

$$\begin{aligned} u_{N-p-q-1} &= (-1)^j u_{p+q} \ , \\ v_{N-p-q-1} &= (-1)^k v_{p+q} \ , \end{aligned} \tag{4.7}$$

$$\begin{aligned} u_{\frac{N-p-1}{2}} &= (-1)^j u_{\frac{N+p-1}{2}} \ , \\ v_{\frac{N-p-1}{2}} &= (-1)^k v_{\frac{N+p-1}{2}} \ . \end{aligned} \tag{4.8}$$

4.2. Diagonals of a DCT-II-transformed symmetric Toeplitz matrix

Let $\mathbf{R}_x$ be a symmetric Toeplitz matrix representing the autocorrelation of a stationary random process, and let $r_z(j,k)$ be the elements of the transformed matrix $\mathbf{R}_z = \mathbf{DWR}_x\mathbf{W}^T\mathbf{D}^T$, with $\mathbf{DW}$ the DCT-II matrix. Then, the following holds.

Theorem 4.1. *Elements $r_z(j,k)$ are zero for odd $j+k$*

Proof. We can also express the first sum in (4.3) as

$$\begin{cases} \displaystyle\sum_{q=0}^{\frac{N-2}{2}} u_q v_q + u_{N-q-1} v_{N-q-1} & \\ \qquad\qquad\qquad\qquad \text{for even } N & \\ \displaystyle\cos\frac{\pi}{2} + \sum_{q=0}^{\frac{N-3}{2}} u_q v_q + u_{N-q-1} v_{N-q-1} & \\ \qquad\qquad\qquad\qquad \text{for odd } N & \end{cases} . \tag{4.9}$$

Substituting the identities (4.5) and (4.6) in (4.9), we find that for odd $j+k$

$$\sum_{q=0}^{N-1} u_q v_q = 0 \ . \tag{4.10}$$

Next, for the terms related to the off-diagonal elements of $\mathbf{R}_x$, the rightmost sum in (4.3) can be expressed as

$$\begin{cases} \displaystyle\sum_{q=0}^{\frac{N-p-2}{2}} W(q,q+p) + W(N-p-q-1, N-q-1) & \\ \qquad\qquad\qquad\qquad \text{for even } N-p & \\ W\left(\frac{N-p-1}{2}, \frac{N+p-1}{2}\right) + \cdots & \\ \displaystyle + \sum_{q=0}^{\frac{N-p-3}{2}} W(q,q+p) + W(N-p-q-1, N-q-1) & \\ \qquad\qquad\qquad\qquad \text{for odd } N-p & \end{cases} , \tag{4.11}$$

where $W(m,n) = u_m v_n + u_n v_m$, for integer m and n.

Using (4.8) we can easily find that for odd $j+k$

$$W\left(\frac{N-p-1}{2}, \frac{N+p-1}{2}\right) = 0, \tag{4.12}$$

and from (4.7) that

$$W(N-p-q-1, N-q-1) = -W(q,q+p) \ . \tag{4.13}$$

Substituting (4.12) and (4.13) in (4.11), shows that

$$\sum_{q=0}^{N-p-1} u_q v_{q+p} + u_{q+p} v_q = 0 \; . \tag{4.14}$$

Finally, by substituting (4.10) and (4.14) in (4.3), it follows that

$$r_z(j,k) = 0 \; , \tag{4.15}$$

for odd $j + k$. □

5. Conclusion

A useful and interesting property of the DCT has been described. The existence of zeros in a DCT-transformed symmetric Toeplitz matrix, suggests the possibility of improving current signal processing systems, in which DCT operates as a key element for stationary signals.

Future research can take advantage of this property, for example in more efficient implementations of fast algorithms for DCT, DFT and other related transforms, for weakly stationary processes. Possible applications include image and video coding, speech analysis and coding, pattern recognition, image and video coding, adaptive filtering and data compression.

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Scattering of a Plane Wave by "Hard-Soft" Wedges

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Abstract. We continue to investigate a nonstationary scattering by wedges [1]–[4]. In this paper we consider a nonstationary scattering of plane waves by a "*hard-soft*" wedge. We give a method for the proof of the existence and uniqueness of solution to the corresponding DN-Cauchy problem in appropriate functional spaces. We show also that the Limiting Amplitude Principle holds.

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Keywords. Diffraction by Wedge, Limiting Amplitude Principle.

1. Introduction

Is this paper we investigate the scattering of a plane wave on a wedge $W := \{y = (y_1, y_2) \in \mathbb{R}^2 : \ y_1 = \rho\cos\theta,\ y_2 = \rho\sin\theta,\ \rho > 0,\ 0 < \theta < \phi\}$ of a magnitude

$$0 < \phi < \pi,\ \Phi = 2\pi - \phi. \tag{1.1}$$

We consider an incident plane wave $u_{\text{in}}(y,t)$ of the form

$$u_{\text{in}}(y,t) = e^{i(k_0\cdot y - \omega_0 t)} f(t - n_0 \cdot y) \text{ for } t \in \mathbb{R} \text{ and } y \in Q := \mathbb{R}^2 \setminus W. \tag{1.2}$$

Here $\omega_0 > 0$ is a frequency of the wave $u_{\text{in}}(y,t)$, while the unit vector and the corresponding wave vector are

$$n_0 = (\cos\alpha, \sin\alpha) \in \mathbb{R}^2,\ k_0 = \omega_0 n_0 \in \mathbb{R}^2 \tag{1.3}$$

respectively. The function f is the profile of the wave. Suppose that $f \in C^\infty(\mathbb{R})$ and for some $\tau_0 > 0$,

$$f(s) = \begin{cases} 0, & s \le 0, \\ 1, & s \ge \tau_0. \end{cases} \tag{1.4}$$

In this case the front of the wave $u_{in}(y,t)$ in the moment t is the line in $\mathbb{R}^2$, $\{y : t - n_0 \cdot y = 0\}$. Let us suppose that for $t < 0$ the front does not intersect the wedge W and at the moment $t = 0$ the front intersect the boundary of W at the vertex of W. It is provided by the following condition: $\max\{\phi-\frac{\pi}{2}\ ,\ 0\}<\alpha<\min\{\frac{\pi}{2}\ ,\phi\}$. Split $\partial Q=Q_1\cup Q_2$ where $Q_1:=\{y=(y_1,y_2)\in\partial Q: y_2=0\}$ and $Q_2:=\{y=(y_1,y_2)\in\partial Q: y_1=\rho\cos\phi, y_2=\rho\sin\phi,\ \rho>0\}$. We will consider the scattering of the wave (1.2) on a wedge of the "*hard-soft*" type. Mathematically this means that we will consider the mixed DN-problem for the wave equation with the Dirichlet condition on a side of the wedge, say Q_2 and the Neumann condition on the other side Q_1 of the wedge:

$$\left\{\begin{array}{rl}\Box u(y,t)=0, & y\in Q\\ \partial_{y_2}u(y,t)=0, & y\in Q_1\\ u(y,t)=0, & y\in Q_2\end{array}\right| \ t\in\mathbb{R}, \tag{1.5}$$

where $\Box=\partial_t^2-\Delta$. We include the ingoing wave $u_{\rm in}$ in the statement of the problem through the initial condition

$$u(y,t)=u_{\rm in}(y,t),\ y\in Q,\ t<0. \tag{1.6}$$

We derive for the first time the Sommerfeld-type representation for the solution $u(y,t)$ of the DN-Problem of nonstationary diffraction. We obtain the uniqueness and the existence of the solution and we prove the Limiting Amplitude Principle, namely, we prove that $u(y,t)\sim e^{-i\omega_0 t}u_\infty(y),\ t\to\infty$, where $u_\infty(y)$ is the Limiting Amplitude. Also we prove a Sommerfeld-type representation for the limiting amplitude. The Sommerfeld representation plays a key role in the scattering by wedges, since it gives a representation of the solution as a superposition of plane waves. Our progress in the justification of the Sommerfeld representation for the DN-Problem is based on the general method of complex characteristics developed in [6]–[9].

2. Definitions and main theorems

1. Let $u(t)\in C(\mathbb{R})$, such that $u(t)=0$ for $t\le T$ and $|u(t)|\le C(1+|t|)^N$ for some $C,N\in\mathbb{R}$. We denote its Fourier-Laplace transform in time as

$$\hat u(\omega):=F_{t\to\omega}[u](\omega):=\int_{-\infty}^{\infty}e^{i\omega t}u(t)dt=\int_T^{\infty}e^{i\omega t}u(t)dt,\ \operatorname{Im}\omega>0. \tag{2.1}$$

Let us denote $\mathbb{C}^+:=\{\omega\in\mathbb{C}:\ \operatorname{Im}\ \omega>0\}$. Obviously, $\hat u(\omega)$ is an analytic function in $\omega\in\mathbb{C}^+$.

2. We will also use the real and complex Fourier transforms in the space variables. Let us consider $u(x)\in C_0^\infty(\mathbb{R}^n),\ n=1,2$. We denote

$$\tilde u(\xi):=F_{x\to\xi}[u](\xi):=\int_{\mathbb{R}^n}e^{i\xi x}u(x)dx,\ \xi\in\mathbb{R}^n. \tag{2.2}$$

We will use similar notations for tempered distributions $u\in S'(\mathbb{R}^n)$. By the Paley-Wiener Theorem [5, Teorema I.5.2], the distribution $\tilde u(\xi)$ has an analytic continuation to the set $K_+^n:=\{z\in\mathbb{C}^n:\ \operatorname{Im}z\in K_+^n\}$, if supp $u\subset K_+^n:=\{x\in\mathbb{R}^n:$

$x_i \geq 0,\ i = 1, \ldots, n\}$. We will denote this analytic continuation by the same symbol $\tilde{u}(z)$ and call the complex Fourier transforms of u. For the regular tempered functions $u(x)$ with support in K_+^n its complex Fourier transforms are expressed as

$$\tilde{u}(z) = \int_{K_+^n} e^{i\langle x,z\rangle} u(x)dx,\ z \in K_+^n. \tag{2.3}$$

3. We denote by $\dot{\overline{Q}} \equiv \overline{Q} \setminus \{0\}$, $\{y\} := |y|/(1+|y|),\ y \in \mathbb{R}^2$ or $y \in \mathbb{R}$.

Definition 2.1.

i) $E_\varepsilon(\Omega)$ is the space of functions $u(y) \in C(\overline{Q}) \cap C^1(\dot{\overline{Q}})$ with the finite norm $|u|_\varepsilon = \sup_{y\in\overline{Q}} |u(y)| + \sup_{y\in\dot{\overline{Q}}} \{y\}^\varepsilon |\nabla u(y)| < \infty,\ \varepsilon \geq 0$

ii) $\mathcal{E}_{\varepsilon,N}(\Omega)$ is the space of functions $u(y,t) \in C^\infty(\dot{\overline{Q}} \times \overline{\mathbb{R}}_+) \cap C(\overline{Q} \times \overline{\mathbb{R}}_+)$, $t \geq 0$ and $y \in \overline{Q}$, with the finite norm

$$\|u\|_{\varepsilon,N} := \sup_{t\geq 0} \Big[\sup_{y\in\overline{Q}} |u(y,t)| + \sup_{y\in\dot{\overline{Q}}} (1+t)^{-N} \{y\}^\varepsilon |\nabla_y u(y,t)| \Big] < \infty,\ N \geq 0. \tag{2.4}$$

Remark 2.1. Obviously, if $u(y,t) \in \mathcal{E}_{\epsilon,N}$, then $\hat{u}(y,\omega) \in E_\epsilon$ for $\omega \in \mathbb{C}^+$.

Definition 2.2. We introduce the following Sommerfeld-type contours, in the form:

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \tag{2.5}$$

where $\mathcal{C}_1 = \Big\{a_1 - \frac{i\pi}{2} : a_1 \geq 1\Big\} \cup \Big\{1 + ib_1 : -\frac{5\pi}{2} \leq b_1 \leq -\frac{\pi}{2}\Big\} \cup \Big\{a_1 - \frac{5i\pi}{2} : a_1 \geq 1\Big\}$.

The contour $\mathcal{C}_2$ is the reflection of $\mathcal{C}_1$ with respect to point $-\frac{3\pi}{2}$. We choose the orientation of the contour, in the sense counter clock-wise (see Figure 1).

The main results of this paper are following theorems. Let

$$g(\omega) = \widehat{f}(\omega - \omega_0) = \int_0^\infty e^{i(\omega-\omega_0)s} f(s)ds,\ \omega \in \mathbb{C}^+ \tag{2.6}$$

where

$$g(\omega) = \frac{\hat{g}_1(\omega)}{\omega - \omega_0},\ \omega \in \mathbb{C}^+,\ \hat{g}_1(\omega) = i\hat{h}(\omega - \omega_0),\ \omega \in \mathbb{C} \tag{2.7}$$

with $h = f'$. Denote

$$H_1(\mu,\alpha,\Phi) := \frac{1}{\sinh[(\mu-\mu_1)q]} + \frac{1}{\sinh[(\mu-\mu_1^*)q]},\ q = \frac{\pi}{2\Phi} \tag{2.8}$$

where

$$\mu_1 = -\frac{i\pi}{2} + i\alpha,\ \mu_1^* = -\mu_1 + i\pi \tag{2.9}$$

Theorem 2.3. *Let the incident wave profile be a smooth function of the type* (1.4), *$u(y,t)$ be a solution to the scattering problem* (1.5), (1.6) *and $u(y,t) \in \mathcal{E}_{\varepsilon,\mathrm{N}}$ with $\varepsilon \in [0,1)$ and $N \geq 0$. Then the solution $u(y,t)$*

i) *is unique*

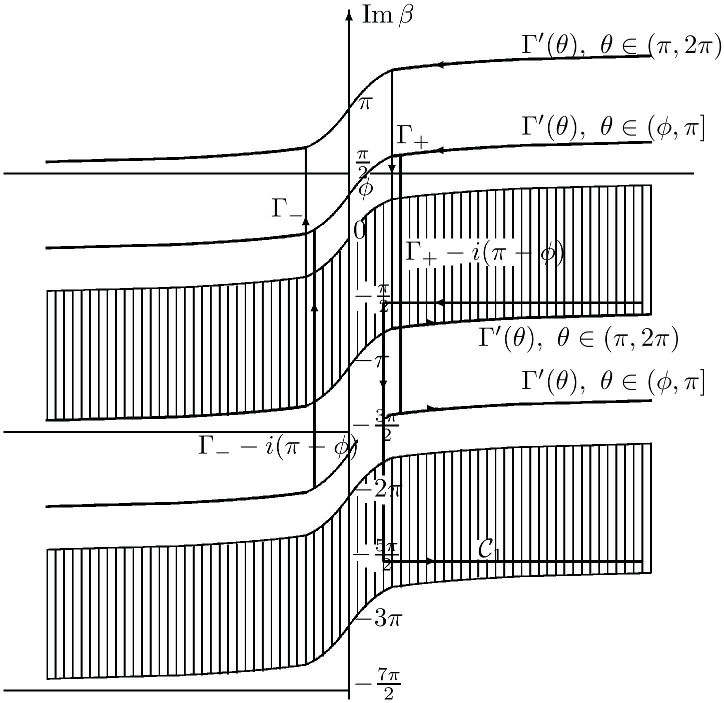

FIGURE 1

ii) *is given by the inverse Fourier transform*

$$u(y,t) = F^{-1}_{\omega\to t}[\hat{u}(y,\omega)],\ t \geq 0,\ (\rho,\theta) \in \overline{Q}, \tag{2.10}$$

where $\hat{u}(y,\omega)$ *in the polar coordinate* $y = \rho e^{i\theta}$ *is the Sommerfeld-type integral*

$$\hat{u}(y,\omega) = \frac{ig(\omega)}{4\Phi} \int\limits_{\mathcal{C}} e^{-\rho\omega\sinh\mu} H_1(\mu + i\theta, \alpha, \Phi)d\mu,\ \rho \geq 0,\ \phi \leq \theta \leq 2\pi,\ \omega \in \mathbb{C}^+. \tag{2.11}$$

Let

$$u_\infty(y) := \frac{i}{4\Phi} \int\limits_{\mathcal{C}} e^{-\omega_0\rho\sinh\beta} H_1(\beta + i\theta)d\beta \tag{2.12}$$

Theorem 2.4.

i) *Let the incident wave profile* $f(s)$ *be a smooth function of the type* (1.4). *Then the function* $u(y,t)$*, defined by* (2.10)*, belongs to the space* $\mathcal{E}_{\epsilon,\mathrm{N}}$ *with* $\epsilon = N = 1 - \frac{\pi}{2\Phi}$*, and is a solution to the scattering problem* (1.5), (1.6).

ii) *The Limiting Amplitude Principle holds: for any* $\rho_0 > 0$,

$$u(y,t) - e^{-i\omega_0 t}u_\infty(\rho,\theta) \to 0, \quad t \to \infty$$

uniformly for $\rho \in [0,\rho_0]$ *and* $\theta \in [\phi, 2\pi]$.

3. Difference equation

3.1. Reduction to stationary problem

Let us consider problem (1.5), (1.6). We apply the complex Forier transform (2.1) with respect to time t to equation (1.5) to get the Helmholtz stationary equation with a parameter. First, define the scattered wave. Let $u(y,t) \in \mathcal{E}_{\epsilon,\mathrm{N}}$ be a solution of the problem (1.5), (1.6) for some $\epsilon > 0, N > 0$. Define the scattered wave as:

$$u_s(y,t) := u(y,t) - u_{\mathrm{in}}(y,t),\ t \in \mathbb{R},\ y \in Q. \tag{3.1}$$

Then (1.6) implies

$$u_s(y,t) \equiv 0,\ t \leq 0,\ y \in Q. \tag{3.2}$$

Furthermore, $u_s(y,t)$ is a solution to the problem

$$\left\{\begin{array}{rll} \Box u_s(y,t) = 0, & y \in Q & \\ \partial_{y_2} u_s(y,t) = -\partial_{y_2} u_{in}(y,t), & y \in Q_1 & t > 0, \\ u_s(y,t) = -u_{in}(y,t), & y \in Q_2 & \end{array}\right. \tag{3.3}$$

$$\{u_s(y,0) = 0,\ \dot{u}_s(y,0) = 0|,\quad y \in Q. \tag{3.4}$$

Remark 3.1. Let us note that $u_{\mathrm{in}}(y,t) \in \mathcal{E}_{0,0}$. Therefore, the condition $u(y,t) \in \mathcal{E}_{\varepsilon,N}$ is equivalent to condition $u_s(y,t) \in \mathcal{E}_{\varepsilon,N}$. Hence, the problems (1.5), (1.6) and (3.3), (3.4) are equivalent in $\mathcal{E}_{\varepsilon,N}$.

Let us apply the complex Fourier transform in time to problem (3.3). First, we apply the complex Fourier transform to (1.2) with respect to t. Making the change of variable $\xi = t - n_0 \cdot y$ and using (1.3), (2.6) and the polar coordinates of y we get

$$\hat{u}_{in}(y,\omega) = g(\omega) e^{i\omega\rho\cos(\theta-\alpha)},\ y \in \mathbb{R}^2,\ \omega \in \mathbb{C}^+. \tag{3.5}$$

Hence

$$\partial_{y_2} \widehat{u}_{in}(y,\omega)\Big|_{y\in Q_1} = i\omega g(\omega) \sin\alpha e^{i\omega y_1 \cos\alpha},\ \widehat{u}_{in}(y,\omega)\Big|_{y\in Q_2} = g(\omega) e^{-i\omega y_2 \frac{\cos(\alpha+\Phi)}{\sin\Phi}}.$$

Therefore, the scattering problem (3.3) is reduced to the following stationary problem.

Lemma 3.2. *Let $u_s(y,t) \in \mathcal{E}_{\varepsilon,N}$ be a solution to problem* (3.3)*; then*

i) *The function $\hat{u}_s(y,\omega)$ is a solution to the following boundary value problem with a parameter $\omega \in \mathbb{C}^+$,*

$$\left\{\begin{array}{ll} (-\Delta - \omega^2)\widehat{u}_s(y,\omega) = 0, & y \in Q \\ \partial_{y_2}\widehat{u}_s(y,\omega) = -i\omega g(\omega) \sin\alpha e^{i\omega y_1\cos\alpha}, & y \in Q_1 \\ \widehat{u}_s(y,\omega) = -g(\omega) e^{-i\omega y_2 \frac{\cos(\alpha+\Phi)}{\sin\Phi}}, & y \in Q_2 \end{array}\right| \quad \omega \in \mathbb{C}^+. \tag{3.6}$$

ii) *The function $\hat{u}_s(\cdot,\omega) \in E_\epsilon$ for $\omega \in \mathbb{C}^+$.*

3.2. Reduction to the problem in a plane

In this section we reduce the problem (3.6) in the angle, to a problem in the plane. Suppose that $\widehat{u}_s(y,\omega) \in E_\varepsilon$ satisfies system (3.6), $\omega \in \mathbb{C}^+$. Let us change the variables $(x_1,x_2) = \mathcal{L}(y)$, where transformation $\mathcal{L}$ maps the angle Q onto $K := \{(x_1,x_2) : \ x_1 < 0 \text{ or } x_2 < 0\}$: $x_1 = y_1 + y_2 \cot\Phi$; $x_2 = -\dfrac{y_2}{\sin\Phi}$. We find the new form of system (3.6) in the coordinates (x_1,x_2). Let $v(x_1,x_2,\omega)$ be the function defined for:

$$v(x_1,x_2,\omega) = \widehat{u}_s(\mathcal{L}^{-1}(x_1,x_2),\omega),\ \omega \in \mathbb{C}^+,\ (x_1,x_2) \in K. \tag{3.7}$$

The function v depends on the parameter ω, nevertheless, in future, we will write simply $v(x)$ instead of $v(x_1,x_2,\omega)$. It is easy to see that system (3.6) for the function (3.7) takes the form:

$$\begin{cases} \mathcal{H}(D)v(x) := \\ \quad = \left(-\dfrac{1}{\sin^2\Phi}\left[\Delta_x - 2\ \cos\Phi\ \partial^2_{x_1x_2}\right] - \omega^2\right) v(x) = 0, & x \in K \\ \cos\Phi\partial_{x_1} v(x_1,0) - \partial_{x_2} v(x_1,0) \\ \quad = -i\omega g(\omega)\sin\alpha\ \sin\Phi\ e^{i\omega x_1\cos\alpha}, & x_1 > 0 \\ v(0,x_2) = -g(\omega)e^{i\omega x_2\cos(\alpha+\Phi)}, & x_2 > 0 \end{cases} \tag{3.8}$$

where $D = (i\partial_{x_1}, i\partial_{x_2})$. Obviously $u \in E_\epsilon(\Omega)$. The function $v \in E_\varepsilon(K)$ which is the space of the functions $v(x) \in C(\overline{K}) \cap C^1(\dot{\overline{K}})$ ($\dot{\overline{K}} := \overline{K} \setminus \{0\}$) such that

$$|v|_\varepsilon = \sup_{x\in\overline{K}} |v(x)| + \sup_{x\in\dot{K}} \{x\}^\varepsilon |\nabla_x v| < \infty. \tag{3.9}$$

We suppose that

$$\varepsilon < 1. \tag{3.10}$$

Then if $v \in E_\varepsilon$, the function v possesses the following Neumann data on $\partial\dot{K}$:

$$v^1_1(x_1) := \partial_{x_2} v(x_1,0),\ x_1 > 0; \qquad v^1_2(x_2) := \partial_{x_1} v(0,x_2),\ x_2 > 0 \tag{3.11}$$

and the following Dirichlet data on ∂K:

$$v^0_1(x_1) := v(x_1,0),\ x_1 > 0; \qquad v^0_2(x_2) := v(0,x_2),\ x_2 > 0 \tag{3.12}$$

Let us extend $v^\beta_l(x_l)$ by zero for $x_l < 0$. Then, by (3.9),

$$|v^1_l(x_l)| \le C\{x_l\}^{-\varepsilon},\ x_l \in \mathbb{R}\setminus\{0\},\ |v^0_l(x_l)| \le C_0,\ x_l \in \mathbb{R}\big|\ \ l = 1,2. \tag{3.13}$$

Therefore, $v^\beta_l(x_l) \in S'(\mathbb{R}),\ l = 1,2,\ \beta = 0,1;$ supp $v^\beta_l(x_l) \subset \overline{\mathbb{R}^+}$ and

$$v^\beta_l(x_l) \in L^1_{loc}(\mathbb{R}) \tag{3.14}$$

by (3.10).

We extend $v(x)$ by zero outside of K and denote

$$v_0(x) = \begin{cases} v(x), & x \in \overline{K} \\ 0, & x \notin \overline{K}. \end{cases}$$

Then $v_0(x)$ determines a regular distribution in $\mathbb{R}^2$ by the definition of the space E_ε, since $\varepsilon < 1$. Let $v(x) \in E_\varepsilon$ with $\varepsilon \in (0,1)$ be a solution to (3.8). Then, in the sense of distributions,

$$\mathcal{H}(D)v_0(x) = d_0(x),\ x \in \mathbb{R}^2, \tag{3.15}$$

where $d_0(x)$ is the distribution of the form

$$\begin{aligned} d_0(x) = \frac{1}{\sin^2\Phi}\Big[&\delta(x_2)v_1^1(x_1) + \delta'(x_2)v_1^0(x_1) + \delta(x_1)v_2^1(x_2) + \delta'(x_1)v_2^0(x_2) \\ &- 2\cos\Phi\ \delta(x_2)\partial_{x_1}v_1^0(x_1) - 2\cos\Phi\ \delta(x_1)\partial_{x_2}v_2^0(x_2) - 2\cos\Phi v(0)\delta(x)\Big]. \end{aligned} \tag{3.16}$$

Now we establish relation between the Cauchy data, generated by boundary conditions.

Proposition 3.3. *Let $v(x) \in E_\varepsilon(K)$ be a solution of* (3.8)*. Then the Cauchy data v_l^β for $l = 1,2$ and $\beta = 0,1$ defined in* (3.11), (3.12) *(and extended by zero in $x_l < 0$) satisfy the conditions:*

$$\begin{cases} \cos\Phi\big[\partial_{x_1}v_1^0(x_1) - \delta(x_1)v(0)\big] - v_1^1(x_1) \\ \qquad = -i\omega g(\omega)\sin\alpha\sin\Phi\ \theta(x_1)e^{i\omega x_1\cos\alpha}, & x_1 \in \mathbb{R}, \\ v_2^0(x_2) = -g(\omega)\theta(x_2)e^{i\omega x_2\cos(\alpha+\Phi)}, & x_2 \in \mathbb{R}. \end{cases} \tag{3.17}$$

Thus, we have reduced the system (3.8) defined in K, to the system (3.15), (3.17), where now two Cauchy data $v_2^0(x_2)$ and one of $v_1^0(x_1)$, $v_1^1(x_1)$ are known functions, by (3.17). In the next section, we reduce this system to the system with the Fourier transform of v_l^β for $l = 1,2$, $\beta = 0,1$. Thereafter we will find the remaining Cauchy data of v_0.

3.3. Fourier Transform

Let us apply the Fourier transform (2.2) to equation (3.15). We obtain

$$\mathcal{H}(\omega,\xi)\tilde{v}_0(\xi) \equiv \left[\frac{1}{\sin^2\Phi}(\xi_1^2 + \xi_2^2 - 2\cos\Phi\ \xi_1\xi_2) - \omega^2\right]\tilde{v}_0(\xi) = \tilde{d}_0(\omega,\xi),\ \xi \in \mathbb{R}^2, \tag{3.18}$$

where $\tilde{v}_0(\xi)$ and $\tilde{d}_0(\xi)$ denote the Fourier transform (2.2) of the tempered distributions v_0 and d_0. The identity (3.18) is also understood in the sense of distributions. Formula (3.16) implies that

$$\begin{aligned} \tilde{d}_0(\omega,\xi) = \frac{1}{\sin^2\Phi}\Big[&\tilde{v}_1^1(\xi_1) - \tilde{v}_1^0(\xi_1)(i\xi_2 - 2i\xi_1\cos\Phi) + \tilde{v}_2^1(\xi_2) \\ &- \tilde{v}_2^0(\xi_2)(i\xi_1 - 2i\xi_2\cos\Phi) - 2\cos\Phi v(0)\Big]. \end{aligned} \tag{3.19}$$

The identity (3.18) allows us to express the solution as

$$\tilde{v}_0(\xi) = \frac{\tilde{d}_0(\omega,\xi)}{\mathcal{H}(\omega,\xi)},\ \xi \in \mathbb{R}^2 \tag{3.20}$$

since $\mathcal{H}(\omega,\xi)\neq 0$ for $\xi\in\mathbb{R}^2$ and $\omega\in\mathbb{C}^+$. It remains to determine the unknown functions $\tilde{v}_2^1(\xi)$ (the Fourier transform of the Neumann data $v_2^1(x_1)$) and one of the functions $\tilde{v}_1^0(\xi_1)$ or $\tilde{v}_1^1(\xi_1)$ (the Fourier transform of the Dirichlet data $\tilde{v}_l^0$ de (3.12)). For this, we will use the equations (3.17) and (3.18). We use that functions $\tilde{v}_l^1(\xi)$ satisfy a connection equation which is an algebraic relation on the Riemann surface of the complex characteristics of the Helmholtz operator $\mathcal{H}$ (see [3]). We will find a particular solution to this connection equation, reducing it to a difference equation. Then we will prove that this particular solution satisfies a certain growth estimate on the Riemann surface. Any solution from E_ε satisfies these growth estimates. This allows us to identify the particular solution with the unique solution from the space E_ε. This identification leads to the uniqueness and the Sommerfeld-type representation. We note that (3.13)–(3.14) imply that $\widetilde{v}_l^\beta(\xi_l)$, $\xi_l\in\mathbb{R}$, admit the analytical continuations to $\mathbb{C}^+$ and these analytical continuations are the complex Fourier transform of the functions $v_l^\beta(x_l)$ in the sense (2.3), for $n=1$. Thus, the complex Fourier transform of the system (3.17), takes the form:

$$\begin{cases}\cos\Phi[-iz_1\tilde{v}_1^0(z_1)-v(0)]-\tilde{v}_1^1(z_1)=\dfrac{\omega g(\omega)\sin\alpha\sin\Phi}{z_1+\omega\cos\alpha}, & \operatorname{Im}\ z_1>0,\\[2ex] \tilde{v}_2^0(z_2)=\dfrac{-ig(\omega)}{z_2+\omega\cos(\alpha+\Phi)}, & \operatorname{Im}\ z_2>0.\end{cases}\tag{3.21}$$

3.4. Riemann surface

To formulate the connection equation, we recall some notations from [8], [9]. Let us denote by $V=V(\omega)$ the Riemann surface $V=\{(z_1,z_2)\in\mathbb{C}^2 \mid z_1^2+z_2^2-2\cos\Phi z_1z_2-\omega^2\sin^2\Phi=0\}$. The surface V has a universal covering surface $\check{V}\cong\mathbb{C}$ with the projection $p:\check{V}\to V$ defined by

$$p:\mu\mapsto(z_1,z_2),\quad \begin{cases}z_1=z_1(\mu):=-i\omega\sinh\mu\\ z_2=z_2(\mu):=-i\omega\sinh(\mu+i\Phi).\end{cases}\tag{3.22}$$

Let us define $\check{V}_l^+$ for $l=1$ resp. $l=2$ as the connected component of the set $\{\mu\in\mathbb{C}:\operatorname{Im} z_l(\mu)>0\}$ which contains the point $\mu=\frac{i\pi}{2}$ resp. $\mu=i\left(\frac{\pi}{2}-\Phi\right)$ (see hence and in the follows Figure 2 which corresponds to the case $\operatorname{Re}\omega>0$). Then $\partial\check{V}_l^+=\check{\Gamma}_l^+\cup\check{\Gamma}_l^-$, where

$$\begin{aligned}\check{\Gamma}_1^\mp&:=\{\mu\in\mathbb{C}:\operatorname{Im} z_1(\mu)=0,\ 0,i\pi\in\check{\Gamma}_1^\mp\ \text{respectively}\},\\ \check{\Gamma}_2^\mp&=\{\mu\in\mathbb{C}:\operatorname{Im} z_2(\mu)=0,\ \pi-i\Phi,-i\Phi\in\check{\Gamma}_2^\mp\ \text{respectively}\}.\end{aligned}\tag{3.23}$$

We check that $\check{\Gamma}_1^-=\left\{\mu=(\mu_1+i\mu_2):\mu_{1,2}\in\mathbb{R},\ \mu_2=\arctan\left(\frac{\omega_1}{\omega_2}\tanh\mu_1\right)\right\}$ with the gauge $\arctan 0=0$. The same representation holds for $\check{\Gamma}_1^+$ with the gauge $\arctan 0=\pi$. Therefore, the contour $\check{\Gamma}_1^+$ is the translation of $\check{\Gamma}_1^-$ by the vector πi: $\check{\Gamma}_1^+=\check{\Gamma}_1^-+\pi i$. Similarly, the contour $\check{\Gamma}_2^-$ is the translation of $\check{\Gamma}_2^+$ by πi, and $\check{\Gamma}_2^+$ is the translation of $\check{\Gamma}_1^-$ by $-i\Phi$. Thus, all the contours (3.23) are identical up to translations. For $\nu\in\mathbb{R}$, let us define the contour $\gamma(\nu)\equiv\check{\Gamma}_1^-+i\nu$. Then

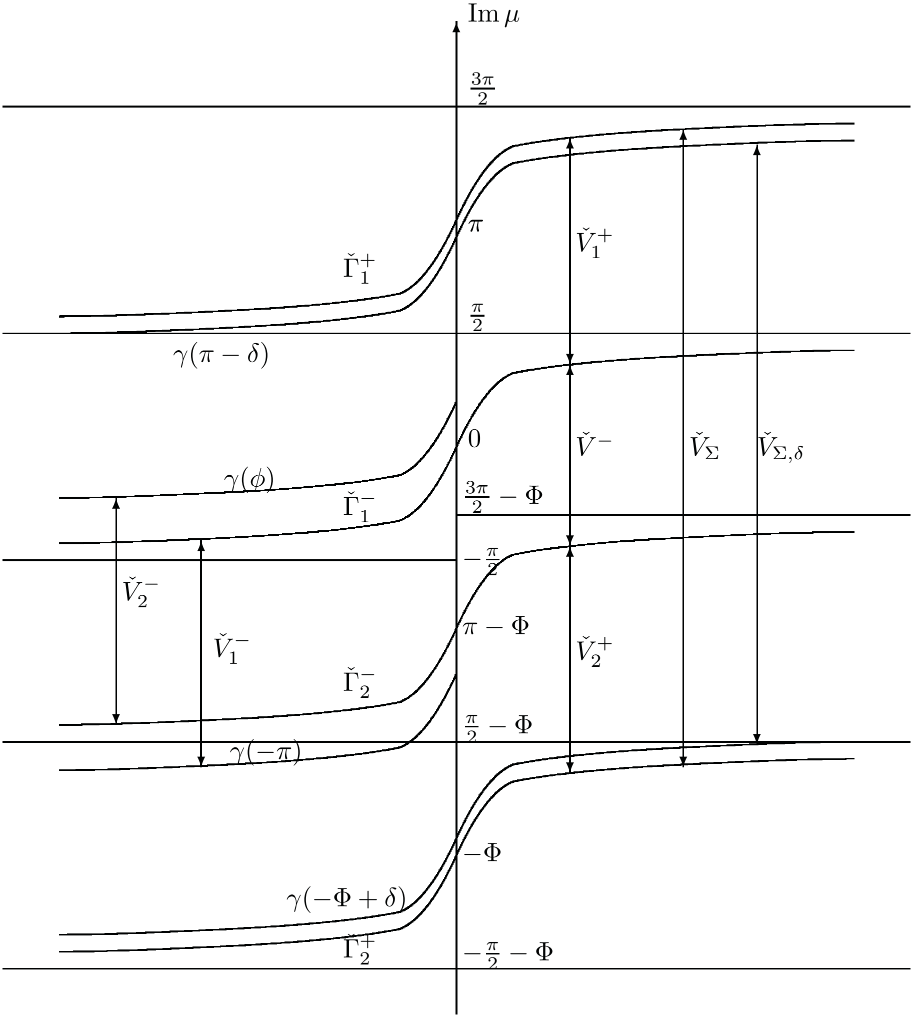

FIGURE 2

the contours (3.23) can be represented in the following form: $\check{\Gamma}_1^- = \gamma(0)$, $\check{\Gamma}_1^+ = \gamma(\pi i)$, $\check{\Gamma}_2^+ = \gamma(-i\Phi)$. Let us define the region $\check{V}_l^-$ for $l = 1, 2$ as the connected component of the set $\{w \in \mathbb{C} : \operatorname{Im} z_l(w) < 0\}$ which contains the point $w = -\frac{i\pi}{2}$. Set $\check{V}^- := \check{V}_1^- \cap \check{V}_2^-$ and $\check{V}_\Sigma := \check{V}_1^+ \cup \overline{\check{V}^-} \cup \check{V}_2^+$. Using the definitions of $\check{V}_l^\pm$, $\check{V}^-$, $\check{V}_\Sigma$ we can represent the regions bounded by the contours $\gamma(\nu)$: $\check{V}_1^+ = \{\mu : \gamma(0) < \mu < \gamma(\pi)\}$, $\check{V}_\Sigma = \{\mu : \gamma(-\Phi) < \mu < \gamma(\pi)\}$. Here the symbol "$<$" means that the point w lies between corresponding curves. Also, we will consider the following subregion $\check{V}_{\Sigma,\delta}$ with a $\delta > 0$: $\check{V}_{\Sigma,\delta} = \{\mu : \gamma(-\Phi + \delta) < \mu < \gamma(\pi - \delta)\}$.

Now we "lift" the functions $\tilde{v}_l^\beta(\mu)$ onto $\check{V}_l^+$ using the covering (3.22). Namely, we denote by $\check{v}_l^\beta(\mu)$ the composition of $\tilde{v}_l^\beta(z_l)$ and $z_l(\mu)$:

$$\check{v}_l^\beta(\mu) = \tilde{v}_l^\beta(z_l(\mu)),\ z_l \in \mathbb{C}^+,\ l = 1, 2,\ \beta = 0, 1. \tag{3.24}$$

The analyticity of the functions $\tilde{v}_l^\beta$ in $\mathbb{C}^+$ implies the analyticity of $\check{v}_l^\beta$ in $\check{V}_l^+$, $l = 1, 2$. We calculate these lifting for the known Dirichlet data of solution. Namely, (3.21), (3.22) and (3.24) give

$$\check{v}_2^0(\mu) = \frac{g(\omega)}{\omega[\sinh(\mu + i\Phi) + i\cos(\alpha + \Phi)]},\ \mu \in \check{V}_2^+.$$

Let $\check{V}_{1,\epsilon} := \{\mu : \gamma(\epsilon) < \mu < \gamma(\pi/2 - \epsilon)\}$ for some $\epsilon > 0$.

Lemma 3.4. *The functions $\check{v}_l^\beta(\mu)$ are analytic in $\check{V}_l^+$ for $l = 1, 2$, $\beta = 0, 1$ and the following estimate holds for some $\epsilon > 0$*

$$|\check{v}_1^0(\mu)| \le Ce^{-|\mathrm{Re}\,\mu|},\ |\mathrm{Re}\,\mu| \ge 1,\ \mu \in \check{V}_{1,\varepsilon}^+. \tag{3.25}$$

3.5. Connection equation and elimination of two Cauchy data

Now we can formulate our basic connection equation [3]. Let us recall that the functions $\check{v}_l^\beta(\mu)$, defined by (3.24), are analytic in the regions $\check{V}_l^+$. By $H(V)$ we denote the set of analytic functions in an open set $V \subset \mathbb{C}$. By $[\check{v}(\mu)]_l$, $l = 1, 2$ we denote the analytic continuation of a function $\check{v}(\mu) \in H(V_l^+)$ to the complex region $\check{V}_\Sigma$ (see Figure 2) if the continuation exists. Let us denote

$$\check{v}_1(\mu) := \check{v}_1^1(\mu) + \omega\sinh(\mu - i\Phi)\check{v}_1^0(\mu) - v(0)\cos\Phi,\ \mu \in \check{V}_1^+ \tag{3.26}$$

and

$$\check{v}_2(\mu) := \check{v}_2^1(\mu) + \omega\sinh(\mu + 2i\Phi)\check{v}_2^0(\mu) - v(0)\cos\Phi,\ \mu \in \check{V}_2^+. \tag{3.27}$$

The following connection equation has been proved in [3], [8].

Proposition 3.5. *Let $v(x) \in E_\varepsilon(K)$ be a solution of (3.8). Then,*

i) *The function $\check{v}_1(\mu)$ admits the analytic continuation from $\check{V}_1^+$ a $\check{V}_\Sigma$, and the function $\check{v}_2(\mu)$ admits the analytic continuation from $\check{V}_2^+$ a $\check{V}_\Sigma$.*

ii) *For the analytic continuations the following connection equation holds:*

$$[\check{v}_1(\mu)]_1 + [\check{v}_2(\mu)]_2 = 0,\ \mu \in \check{V}_\Sigma. \tag{3.28}$$

iii) *The following estimates hold for the analytic continuations:*

$$|[\check{v}_l(\mu)]| \le C_\delta(1 + e^{|\mu|})^q,\ \mu \in \check{V}_{\Sigma,\delta},\ l = 1, 2. \tag{3.29}$$

for any $\delta \in (0, \Phi/2 + \pi/2)$, with a $q \in \mathbb{R}$ depending on $\check{v}_1^1(\mu)$ and $\check{v}_2^1(\mu)$.

We reduce the connection equation (3.28) to a equation that contains only two unknown functions. "Lifting" by the formulas (3.22) the first equation of (3.21) to $\check{V}_1^+$, expressing $\check{v}_1^1(\mu)$ and substituting the obtained expressing into (3.26) we obtain the expressing for $\check{v}_1(\mu)$

$$\check{v}_1(\mu) = -i\omega\sin\Phi\cosh\mu\check{v}_1^0(\mu) - \frac{ig(\omega)\sin\Phi\sin\alpha}{\sinh\mu + i\cos\alpha} - 2v(0)\cos\Phi,\ \mu \in \check{V}_1^+. \tag{3.30}$$

Similarly, substituting into the right-hand side of (3.27) the function $\check{v}_2^0(\mu)$, we obtain

$$\check{v}_2(\mu) = \check{v}_2^1(\mu) + \frac{g(\omega)\sinh(\mu + 2i\Phi)}{\sinh(\mu + i\Phi) + i\cos(\alpha + \Phi)} - v(0)\cos\Phi, \;\; \mu \in \check{V}_2^+.$$

Hence and from (3.28) we deduce the following.

Lemma 3.6. *Let $v(x) \in E_\varepsilon(K)$ be a solution to the problem* (3.8). *Then the functions $\check{v}_1^0(\mu)$ and $\check{v}_2^1(\mu)$ admit meromorphic continuation to $\check{V}_\Sigma$ and satisfy the connection equation:*

$$i\omega \sin\Phi \cosh\mu \check{v}_1^0(\mu) - \check{v}_2^1(\mu) = G(\mu), \;\; \mu \in \check{V}_\Sigma \tag{3.31}$$

with

$$G(\mu) = g(\omega)\left(\frac{\sin\Phi\sin\alpha}{i\sinh\mu - \cos\alpha} + \frac{\sinh(\mu + 2i\Phi)}{\sinh(\mu + i\Phi) + i\cos(\alpha + \Phi)}\right) - 3v(0)\cos\Phi, \quad \mu \in \mathbb{C}.$$

and $\check{v}_1^0(\mu)$, $\check{v}_2^1(\mu)$ determined by (3.24).

3.6. Automorphisms

To reduce the equation (3.31) to the difference equation, we use the automorphism properties of the functions $\check{v}_1^0(\mu)$ y $\check{v}_2^1(\mu)$.

Definition 3.1. For $l = 1, 2$, we define the automorfismos $\check{h}_l : \check{V} \longrightarrow \check{V}$ in the following way: $\check{h}_1$ is the reflection with respect to the point $\frac{i\pi}{2}$, $\check{h}_2$ is the reflection with respect to the point $\frac{i\pi}{2} - i\Phi$, namely: $\check{h}_1(\mu) = -\mu + i\pi$, $\check{h}_2(\mu) = -\mu + i\pi - 2i\Phi$, $\mu \in \mathbb{C}$.

The projections (3.22) satisfy the automorphism conditions $z_l(\check{h}_l(\mu)) = z_l(\mu)$ for $\mu \in \mathbb{C}, l = 1, 2$. Thus, the lifting $\check{v}_1^0(\mu)$, $\check{v}_2^1(\mu)$ also satisfy the same conditions: using these properties and meromorphity of these functions in $\check{V}_\Sigma$ we obtain (see [1], [2]) that $\check{v}_1^0$ admits the meromorphic continuation to $\mathbb{C}$ and satisfies the difference equation

$$\cosh\mu \check{v}_1^0(\mu) + \cosh(\mu + 2i\Phi)\check{v}_1^0(\mu + 2i\Phi) = G_2(\mu), \;\; \mu \in \mathbb{C}. \tag{3.32}$$

where

$$G_2(\mu) = \frac{g(\omega)}{\omega}\left(\frac{\cosh\mu}{\sinh\mu + i\cos\alpha} + \frac{\cosh(\mu + 2i\Phi)}{\sinh(\mu + 2i\Phi) + i\cos\alpha}\right), \;\; \mu \in \mathbb{C}.$$

4. Uniqueness

We start with the following theorem.

Theorem 4.1. *There no exist two functions $\tilde{v}_1^0(z)$, $\tilde{v}_1^{0*}(z)$ satisfying the following conditions*

i) *The functions $\tilde{v}_1^0$, $\tilde{v}_1^{0*}$ defined by* (3.24) *are analytic in $\check{V}_1^+$ admit the meromorphic continuation to $\mathbb{C}$ and satisfy the difference equation* (3.32)

ii) *These functions satisfy the automorphism conditions*

$$\check{v}(-\mu + i\pi) = \check{v}(\mu), \ \mu \in \mathbb{C}. \tag{4.1}$$

iii) *These functions admit the estimate* (3.25)

iv) *The functions* (3.30) *corresponding to* $\check{v}_1^0$ *and* $\check{v}_1^{0*}$ *are meromorphic in* $\mathbb{C}$ *and satisfy* (3.29) *for* $l = 1$.

Corollary 4.1. *Suppose that* $v(x)$ *and* $v^*(x)$ *belong to* $E_\varepsilon(K)$ *and both are solutions to the system* (3.8). *Let* v_l^β *and* $v_l^{\beta*}$ *are defined in* (3.11) *and* (3.12) *by* v *and* v^* *respectively. Then*

$$v_l^\beta = v_l^{\beta*}, \ l = 1, 2, \ \beta = 0, 1 \tag{4.2}$$

Proof. The statement follows from Theorem 4.1, (3.21) and (3.31). □

Proof of Theorem 2.3 (i). Consider two functions $u_s(y,t)$ and $u_s^*(y,t)$ defined by (3.1) and corresponding to u and u^* respectively. Then by Lemma 3.2 $\widehat{u}_s$ and $\widehat{u}_s^*$ belongs to E_ε and satisfy (3.6). The functions $v(x,\omega)$ and $v^*(x,\omega)$ defined by (3.7) and corresponding to $\widehat{u}_s$ and $\widehat{u}_s^*$ satisfy (3.8) and belong to $E_\varepsilon(K)$. Now we are in the situation of Theorem 4.1. All this the identities (4.2) hold. Therefore $v(x,\omega) = v^*(x,\omega)$ by (3.20) and (3.19) for any $\omega \in \mathbb{C}$. This implies that $u_s(y,t) = u_s^*(y,t)$ since these functions are the inverse Fourier-Laplace transform of $u(y,\omega)$ and $u^*(y,\omega)$ which obviously also coincide by (3.7). □

5. Solution of the difference equation

5.1. Meromorphic solution

In this section we construct a meromorphic solution and then an analytic solution for the difference equation (3.32). First, we construct a meromorphic solution of this equation which decreasing as $e^{-|\mathrm{Re}\,\omega|}$ for $\mathrm{Re}\,\omega \to \infty$. Denote

$$\mu_1(k) = \mu_1 + 2k\pi i \text{ and } \mu_2(k) = \mu_2 + 2k\pi i, \ k \in \mathbb{Z} \tag{5.1}$$

where μ_1 is defined in (2.9) and $\mu_2 = -i\pi/2 - i\alpha$. We define

$$\check{w}_1^0(\mu) := \frac{g(\omega)}{\omega}\left(\frac{1}{\sinh\mu + i\cos\alpha}\right), \ \mu \in \mathbb{C}\ . \tag{5.2}$$

Obviously $\check{w}_1^0(\mu)$ is a meromorphic function in $\mathbb{C}$ with poles $\mu_1(k)$ and $\mu_2(k)$ and residues

$$\mathrm{Res}(\check{w}_1^0, \mu_1(k)) = \frac{g(\omega)}{\omega\sin\alpha}, \quad \mathrm{Res}(\check{w}_1^0, \mu_2(k)) = -\frac{g(\omega)}{\omega\sin\alpha}, \quad k \in \mathbb{Z}.$$

Lemma 5.1.

i) *The function* $\check{w}_1^0(\mu)$ *is analytic in* $\check{V}_1^+$.

ii) *It satisfies* (3.32), (4.1).

iii) *The estimate holds* $|\check{w}_1^0(\mu)| \le C(\omega)e^{-|\mathrm{Re}\,\mu|}, \ |\mathrm{Re}\,\mu| \ge 1.$

We define $\check{w}_1(\mu)$ similarly to $\check{v}_1(\mu)$ from (3.30), with the replacement $\check{v}_1^0(\mu)$ by $\check{w}_1^0(\mu)$ from (5.2).

Lemma 5.2.

i) *The function* $\breve{w}_1(\mu)$*, admits the representation*

$$\breve{w}_1(\mu) = -i\omega \sin\Phi \big(\cosh\mu + \sin\alpha\big)\breve{w}_1^0(\mu) - 2v(0)\cos\Phi.$$

ii) *The poles of* $\breve{w}_1(\mu)$ *in* $\mathbb{C}$ *can be only the points* $\mu_1(k)$*,* $\mu_2(k)$*,* $k \in \mathbb{Z}$ *defined in* (5.1)*.*

iii) *The unique pole and residue of* $\breve{w}_1(\mu)$ *in* $\check{V}_\Sigma$ *is* μ_1 *and*

$$\operatorname{Res}(\breve{w}_1, \mu_1) = -2ig(\omega)\sin\Phi.$$

iv) *The function* $\breve{w}_1(\mu)$ *admits the estimate* $|\breve{w}_1(\mu)| \le C(\omega)$*,* $|\operatorname{Re}\mu| \ge 1$*, for some* $C(\omega) > 0$*.*

5.2. Analytic solution

By Lemma 5.1 ii) the function $\breve{w}_1^0(\mu)$ is a particular meromorphic solution to the inhomogeneous difference equation (3.32). However, the corresponding function $\breve{w}_1(\mu)$ from (3.30) is not analytic in $\check{V}_\Sigma$ by Lemma 5.2 iii) that does not correspond to Proposition 3.5. Hence we have to construct a "*correct*" solution to (3.32), (4.1). Let

$$K_2(\mu) := \frac{K_1(\mu)}{\cosh\mu},\ K_1(\mu) := -\frac{\pi g(\omega)}{\omega\Phi}\cdot H_1(\mu,\alpha,\Phi),\ \mu \in \mathbb{C} \tag{5.3}$$

where H_1 is defined by (2.8). We "correct" the function $\breve{w}_1^0$ and define $\breve{v}_1^0$ as

$$\breve{v}_1^0(\mu) := \breve{w}_1^0(\mu) + K_2(\mu). \tag{5.4}$$

First, K_2 is a solution to the homogeneous equation corresponding to (3.32) and satisfies (4.1). Therefore $\breve{v}_1^0$ satisfies (3.32), (4.1) by Lemma 5.1 **ii)**. The corresponding function (3.30) has the form

$$\breve{v}_1(\mu) = \breve{w}_1(\mu) - i\omega\sin\Phi\cdot K_1(\mu). \tag{5.5}$$

In the following lemma we also check that the corresponding function $\breve{v}_1(\mu)$ from (3.30) is analytic in $\check{V}_\Sigma$. The poles of the function $K_1(\mu)$ (coinciding with the poles of H_1) are

$$\beta_k' = \mu_1 + 2\Phi k i,\ \beta_k'' = \check{h}_1(\beta_k') = \mu_1^* + 2i\Phi k,\ k \in \mathbb{Z} \tag{5.6}$$

where μ_1, μ_1^* are defined in (2.9). Since $\Phi > \pi$ only $\beta_0' = \mu_1 \in \check{V}_\Sigma$. Obviously K_1 satisfies the estimate $|K_1(\mu)| \le e^{-\frac{\pi}{2\Phi}|\operatorname{Re}\mu|}$, $|\operatorname{Re}\mu| \ge 1$. From this we obtain the follows statement.

Lemma 5.3.

i) *The function* $\breve{v}_1^0$
- *satisfies* (3.32)*,* (4.1)*.*
- *is meromorphic in* $\mathbb{C}$ *and analytic in* $\check{V}_1^+$*.*
- *satisfies the estimate* $|\breve{v}_1^0(\mu)| \le Ce^{-|\operatorname{Re}\mu|}$*,* $|\operatorname{Re}\mu| \ge 1$

ii) *The corresponding function* $\breve{v}_1(\mu)$ *from* (5.5) *is analytic in* $\overline{\check{V}_\Sigma}$*.*

Corollary 5.4. *If $\widehat{u}_s(y,\omega)\in E_\epsilon$ with $\epsilon\in(0,1)$ is a solution to the problem* (3.6) *for $\omega\in\mathbb{C}^+$, then the function $\check{v}_1(\mu)$ defined by* (3.26) *is the function given by* (5.5).

Proof. Since $\check{v}_1^0(\mu)$ defined in (5.4) satisfies all conditions of the Theorem 4.1, the statement follows from this theorem. □

6. Inverse Fourier transform in time

6.1. Sommerfeld representation for scattered wave

In this section we write the solution $\hat{u}_s$ to problem (3.6) in the Sommerfeld-type representation form (under the assumption of its existence) by means of the solution of the difference equation (3.32). Let $\Gamma(\theta)$ be the contour:

$$\Gamma(\theta)=\begin{cases}\overleftarrow{\gamma(\phi)}\cup\overrightarrow{\gamma(-\Phi)}, & \text{si } \phi<\theta\le\pi\\ \overleftarrow{\gamma(\pi)}\cup\overrightarrow{\gamma(-\pi)}, & \text{si } \pi<\theta<2\pi\end{cases}$$

where ϕ and Φ is defined in (1.1). The orientations of the contour $\Gamma(\theta)$ is shown in the Figure 1. We denote by $-\gamma$ the contour γ with the contrary orientation.

Theorem 6.1 (See Theorem 9.1 and Remark 12.2 from [1]). *If there exists a solution $\widehat{u}_s\in E_\varepsilon$ to the Helmholtz equation in* (3.6)*, then this solution has the form:*

$$\widehat{u}_s(y,\omega)=\frac{1}{4\pi\sin\Phi}\int_{\Gamma(\theta)}e^{-\rho\omega\sinh(\mu-i\theta)}\check{v}_1(\mu)\mathrm{d}\mu, \tag{6.1}$$

for $\rho>0$ and $\phi<\theta<2\pi$, with $\check{v}_1(\mu)$ defined by (3.26).

We give the representation of a solution to the stationary problem (3.6) (with a parameter $\omega\in\mathbb{C}^+$) in the standard form of the Sommerfeld integral. This form is the direct consequence of the representation (6.1), Corollary 5.4, (5.5) and periodicity of functions $\check{w}_1^0(\mu)$, $\check{w}_1(\mu)$ and contours $\Gamma(\theta)$ with the period $2\pi i$.

Theorem 6.2. *If a solution to problem* (3.6) *with $\omega\in\mathbb{C}^+$ exists in the space E_ε with $\varepsilon\in(0,1)$, then it is expressed by the Sommerfeld-type integral*

$$\widehat{u}_s(y,\omega)=\frac{ig(\omega)}{4\Phi}\int_{\Gamma(\theta)}e^{-\rho\omega\sinh(\mu-i\theta)}H_1(\mu,\alpha,\Phi)d\mu,\ \ \phi\le\theta\le2\pi \tag{6.2}$$

where $H_1(\mu,\alpha,\Phi)$ is given by (2.8).

Proof of Theorem 2.3 (ii). Let $u(y,t)\in\mathcal{E}_{\epsilon,N}$, $\varepsilon\in(0,1)$, $N\ge0$ be a solution to the problem (1.5), (1.6). The corresponding scattered wave $u_s(y,t)$ is defined by (3.1). Its Fourier transform in time is expressed by (6.2) according to Lemma 3.2 and Theorem 6.2.
Let us apply the Fourier transform in time to (3.1). First, (1.2) implies (3.5). Therefore, (3.1) implies that $\hat{u}(y,\omega)=\hat{u}_s(y,\omega)+g(\omega)e^{i\omega\rho\cos(\theta-\alpha)}$, $y\in\overline{Q}$, $\omega\in\mathbb{C}^+$. Now (2.11) follows from (6.2), (2.5), (5.6) and the Cauchy Residues Theorem. Theorem 2.3(ii) is proved. □

7. Existence

Let us define the reflected wave $u_r(y,t)$ as

$$u_r(y,t) := \begin{cases} u_{r,1}(y,t), & \phi \le \theta \le \theta_1 \\ 0 & \theta_1 < \theta < \theta_2 \\ u_{r,2}(y,t), & \theta_2 \le \theta \le 2\pi \end{cases}$$

where (ρ,θ) are the polar coordinate of y,

$$\theta_1 := 2\phi - \alpha,\ \ \theta_2 = 2\pi - \alpha \tag{7.1}$$

and

$$u_{r,1}(y,t) = -e^{i(k_1\cdot y - \omega_0 t)} f(t - n_1\cdot y),\ \ u_{r,2}(y,t) = e^{i(k_2\cdot y-\omega_0 t)} f(t - n_2\cdot y), \tag{7.2}$$
$$k_1 = \omega_0 n_1,\ \ n_1 = (\cos\theta_1, \sin\theta_1);\ \ k_2 = \omega_0 n_2,\ \ n_2 = (\cos\theta_2, \sin\theta_2).$$

Introduce the diffracted waves by

$$u_d(y,t) := u_s(y,t) - u_r(y,t). \tag{7.3}$$

7.1. On Sommerfeld-Malyuzhinets integrals

For $\omega \in \overline{\mathbb{C}^+}$ let us denote by $\mathcal{S}(y,\omega)$ the “stationary” Sommerfeld-Malyuzhinets type integral

$$\mathcal{S}(y,\omega) := \frac{i}{4\Phi}\int_{\mathcal{C}} e^{-\omega\rho\sinh\beta} H_1(\beta + i\theta)d\beta,\ \ \rho \ge 0,\ \ \phi \le \theta \le 2\pi \tag{7.4}$$

where H_1 is defined in (2.8) and which absolutely converges (see Lemma 7.2). By Theorem 2.3 ii) if a solution u of the problem (1.5), (1.6) exists, then

$$u(y,t) = F^{-1}_{\omega\to t}[\hat{u}(y,\omega)],\ \ \rho > 0,\ \ \phi < \theta < 2\pi \tag{7.5}$$

where

$$\hat{u}(y,\omega) = g(\omega)\mathcal{S}(y,\omega),\ \ \omega \in \mathbb{C}^+. \tag{7.6}$$

Let us examine the convergence of the integral (7.4) and its derivatives in ω and ρ,θ. First, let us prove the exponential decay of the function H_1. The poles of the function H_1 coincide with the poles of K_1 from (5.3) and are given by (5.6). For $\delta > 0$ denote $\mathbb{C}_\delta := \{\beta \in \mathbb{C} : |\beta - \beta_k| \ge \delta,\ \forall k \in \mathbb{Z}\}$. From (2.8) we obtain

Lemma 7.1. *For any $\delta > 0$ the estimate holds*

$$|H_1(\beta,\alpha,\Phi)| \le C_\delta e^{-\frac{\pi}{2\Phi}|\mathrm{Re}\ \beta|},\ \ \beta \in \mathbb{C}_\delta. \tag{7.7}$$

Further, let us consider $\omega := \omega_1 + i\omega_2$ with $\omega_1 \in \mathbb{R}$ and $\omega_2 \ge 0$ and $\beta = \beta_1 + i\beta_2 \in \mathbb{C}$ with $\beta_{1,2} \in \mathbb{R}$. Denote $\Sigma := Q \times \mathbb{C}^+$.

Lemma 7.2.

i) *The integral* (7.4) *converges absolutely and uniformly for* $(y,\omega) \in \overline{\Sigma}$.

ii) *The function* $\mathcal{S}(y,\omega)$ *is continuous in* $\overline{\Sigma}$.

iii) *The function* $\mathcal{S}(y,\omega)$ *is analytic in* $\omega \in \mathbb{C}^+$ *and smooth in* $(\rho,\theta) \in \dot{\overline{Q}}$.

iv) *The function* $\mathcal{S}(y,\omega) \in C^\infty(\dot{\overline{Q}} \times (\mathbb{R}\setminus\{0\}))$.

Proof. The proof of this Lemma coincides literally with the proof of Lemma 4.2 from [2] with the replacement of the expression $\frac{\pi}{\Phi}$ in [2] by $\frac{\pi}{2\Phi}$. □

Let us *define*

$$\hat{u}_s(y,\omega) = \hat{u}(y,\omega) - \hat{u}_{in}(y,\omega) \tag{7.8}$$

Theorem 7.1. *$\widehat{u}_s(y,\omega)$ is a classical solution to the stationary problem* (3.6) *for $\omega \in \mathbb{C}^+$.*

Proof. It is proved similarly to the Corollary 5.2 from [2]. □

7.2. Incident, reflected and diffracted stationary waves

In this subsections we prove the statement i) of Theorem 2.4 for the function $u(y,t)$ defined by (7.5): namely we prove that the function is a solution to the problem (1.5), (1.6) and belongs to $\mathcal{E}_{\epsilon,N}$ with ϵ, N defined in Theorem 2.4. We deduce (1.6) from (3.2) by the Paley-Wiener theorem, using the estimates of $\hat{u}_s(y,\omega)$ for $\omega \in \mathbb{C}^+$. Note that $\mathcal{S}(y,\omega)$ and $\hat{u}(y,\omega) = \hat{u}_s(y,\omega) + \hat{u}_{in}(y,\omega)$ are not bounded for $\omega \in \mathbb{C}^+$ since $u_{in}(x,t) \not\equiv 0$ for $t < 0$. Therefore, we have to extract first the incident wave from the integral (7.4). The contour $\mathcal{C}$ in the integral (7.4) crosses "bad zones" between $\gamma(-\pi)$ and $\gamma(-2\pi)$, where $\operatorname{Re}(\omega \sinh\beta) < 0$, and the exponent $e^{-\omega\rho\sinh\beta}$ is growing for $\operatorname{Im}\omega \to +\infty$ (see Figure 1). This growing part of the integral just corresponds to the incident wave. To extract the incident wave, we split the function $\mathcal{S}(y,\omega)$ into three summands

$$\mathcal{S} := \mathcal{S}_{in} + \mathcal{S}_s := \mathcal{S}_{in} + \mathcal{S}_r + \mathcal{S}_d \tag{7.9}$$

where for $\theta \neq \theta_{1,2}$

$$\left\{\begin{array}{l} \mathcal{S}_d(y,\omega) := \dfrac{i}{4\Phi}\displaystyle\int_{\mathcal{C}_0} e^{-\omega\rho\sinh\beta} H_1(\beta + i\theta, \alpha, \Phi)\,d\beta \\ \mathcal{S}_{in}(y,\omega) := e^{i\omega\rho\cos(\theta-\alpha)} \\ \mathcal{S}_r(y,\omega) := \begin{cases} -\,e^{i\omega\rho\cos(\theta-\theta_1)} & \phi \le \theta \le \theta_1 \\ 0 & \theta_1 < \theta < \theta_2 \\ e^{i\omega\rho\cos(\theta-\theta_2)}, & \theta_2 \le \theta \le 2\pi \end{cases} \end{array}\right| \; \omega \in \overline{\mathbb{C}^+}, \tag{7.10}$$

θ_1 and θ_2 are defined in (7.1) and $\mathcal{C}_0 := \gamma_1 \cup \gamma_2$, $\gamma_1 := \{\beta_1 - i\pi/2,\ \beta_1 \in \mathbb{R}\}$, $\gamma_2 := \{\beta_1 - 5i\pi/2,\ \beta_1 \in \mathbb{R}\}$. From the estimate (7.7) it follows that the integral in (7.10) converges absolutely for $\omega \in \overline{\mathbb{C}^+}$ and defines a continuous function of $\omega \in \overline{\mathbb{C}^+}$.

Let us note that

$$\hat{u}_{in}(y,\omega) = g(\omega)\mathcal{S}_{in}(y,t) \tag{7.11}$$

by (3.5).

Remark 7.3. Let (7.9) holds. Then (7.8), (7.6) and (7.11) imply that

$$\hat{u}_s = \hat{u} - \hat{u}_{in} = g(\mathcal{S} - \mathcal{S}_{in}) = g\mathcal{S}_s \tag{7.12}$$

We call $\mathcal{S}$, $\mathcal{S}_d$, $\mathcal{S}_{in}$, $\mathcal{S}_r$, $\mathcal{S}_s$ as *densities* of the total, diffracted, incident, reflected and scattered waves respectively. The incident part $\mathcal{S}_{in}(\rho,\theta,\cdot)$ is unbounded in $\mathbb{C}^+$ while the reflected part $\mathcal{S}_r(\rho,\theta,\cdot)$ and the diffracted part $\mathcal{S}_d(\rho,\theta,\cdot)$ are bounded in $\mathbb{C}^+$. Hence, $\hat{u}_s(\rho,\theta,\cdot)$ is bounded in $\mathbb{C}^+$.

7.3. The scattered wave

In this section we prove Theorem 2.4 i), namely that the function (2.10) is a smooth solution to the scattering problem (1.5), (1.6). For this we study the function $u_s(y,t) := F^{-1}_{\omega\to t}[\hat{u}_s(y,\omega)]$. We prove that $u_s(y,t)$ satisfies the system (3.3), (3.2). Note that by Lemma 7.2 ii), iii), formula (7.9) and the definition of $\mathcal{S}_{in}$ in (7.10), $\mathcal{S}_s \in C(\overline{Q}\times\mathbb{R})$ and $\mathcal{S}_s(\cdot,\cdot,\omega)\in C^\infty(\dot{\overline{Q}})$, $\omega\in\mathbb{C}^+$. Let us define the function

$$w_s(y,t) := F^{-1}_{\omega\to t}[\tilde{w}_s(y,\omega)],\ t\in\mathbb{R} \tag{7.13}$$

where the function

$$\tilde{w}_s(y,\omega) := \hat{g}_1(\omega)\mathcal{S}_s(y,\omega),\ \omega\in\overline{\mathbb{C}^+} \tag{7.14}$$

with $\hat{g}_1(\omega)$ given in (2.7).

Lemma 7.4. *For all $(\rho,\theta)\in\overline{Q}$ there exists the inverse Fourier-Laplace transform $w_s(y,t) = F^{-1}_{\omega\to t}[\tilde{w}_s]$ of the function $\tilde{w}_s(y,\omega)$, and*

$$w_s\in C^\infty(\dot{\overline{Q}}\times\mathbb{R})\cap C(\overline{Q}\times\mathbb{R}),\ |w_s(y,t)|\le C,\ t\ge 0, \tag{7.15}$$

$$w_s(y,t) = 0,\ t<0. \tag{7.16}$$

Proposition 7.5.

i) *The function $u_s(y,t)$ admits the following representation*

$$u_s(y,t) = -i\int_0^t e^{i\omega_0(\tau-t)}w_s(\rho,\theta,\tau)d\tau,\ (\rho,\theta)\in\overline{Q};\ t\in\mathbb{R}. \tag{7.17}$$

Furthermore,

$$u_s\in C^\infty(\dot{\overline{Q}}\times\mathbb{R})\cap C(\overline{Q}\times\mathbb{R}) \tag{7.18}$$

and for $(\rho,\theta)\in Q$,

$$u_s(y,t) = 0,\ t<0,\ |u_s(y,t)|\le C(1+t),\ t\ge 0. \tag{7.19}$$

ii) *$u_s(y,t)$ is a solution of system* (3.3) *and satisfies the initial conditions* (3.2).

iii) *$u_s(y,t)$ satisfies the estimate* (2.4) *with $\epsilon = N = 1-\pi/2\Phi$.*

The sketch of the proof. i) The representation (7.17) follows from (7.12), (7.14), (2.7), (7.13) and (7.16). Now (7.18) follows from (7.15) and (7.17). Finally, (7.19) follows from (7.15), (7.16) and (7.17).

ii) The system (3.3), (3.4) holds for u_s in the classical sense since $\widehat{u}_s$ satisfies (3.6) in the classical sense by Theorem 7.1. The identity (3.2) follows from (7.19).

iii) By (7.3) it suffices to prove the estimate for u_d outside the critical directions $\theta = \theta_1,\theta_2$ since the estimate for u_r is trivial. It is proved similarly to Lemma 12.1, Theorem 12.2, Propositions 14.1 and 14.2 from [2] with the replacement of $1-\pi/\Phi$ by $1-\pi/2\Phi$. Moreover the follows key representation is used

$$u_d(y,t) = \frac{ie^{-i\omega_0 t}}{4\Phi}\int_{\mathbb{R}} Z_1(\beta,\theta)h(\beta,\rho,t)d\beta \tag{7.20}$$

with $Z_1(\beta,\theta) := -H_1(-i\pi/2+\beta+i\theta) + H_1(-5i\pi/2+\beta+i\theta)$ and $h(\beta,\rho,t) := f(t-\rho\cosh\beta)e^{i\omega_0\rho\cosh\beta}$, $\beta\in\mathbb{R}$. □

Proof of the Theorem 2.4 i). The statement follows from Remark 3.1, Proposition 7.5 and Definition 2.1 (ii) of $\mathcal{E}_{\epsilon,N}$. □

8. Limiting amplitude principle

Proof of Theorem 2.4 ii). By (3.1) and (7.3) it suffices to prove the Limiting Amplitude Principle only for the diffracted wave u_d since for the u_{in} and u_r it holds by (1.2), (7.2) and (1.4) with the corresponding limiting amplitudes $A_{in}(y) = e^{i\omega_0\rho\cos(\theta-\alpha)}$ and

$$A_r(y) = \begin{cases} -e^{i\omega_0\rho\cos(\theta-\theta_1)}, & \phi\le\theta\le\theta_1\\ 0 & \theta_1<\theta<\theta_2\\ e^{i\omega_0\rho\cos(\theta-\theta_2)}, & \theta_2\le\theta\le 2\pi.\end{cases}$$

By (7.20) the Amplitude $A_d(y,t)$ of $u_d(y,t)$ is expressed as

$$A_d(y,t) := e^{i\omega_0 t}u_d(y,t) = \frac{i}{4\Phi}\int_{\mathbb{R}} Z_1(\beta,\theta)h(\beta,\rho,\theta)d\beta.$$

Similarly to Theorem 15.1 it is proved that $A_d(y,t)\to A_d(y)$, $t\to\infty$ uniformly with the respect to $\rho\le\rho_0$ and $\theta\in[\phi,2\pi]$ where

$$A_d(y) = \frac{i}{4\Phi}\int_{\mathbb{R}} e^{i\omega_0\rho\cosh\beta}Z_1(\beta,\theta)d\beta.$$

It remains only to note that $u_\infty(y) = A_{in}(y)+A_r(y)+A_d(y)$. □

Remark 8.1. It is checked directly that $u_\infty(y)$ given by (2.12) is a solution of the following stationary problem

$$\begin{cases} (\Delta+\omega_0^2)u_\infty(y) = 0, & (\rho,\theta)\in Q\\ \partial_{y_2}u_\infty\big|_{Q_1} = 0 & \\ u_\infty\big|_{Q_2} = 0. & \end{cases}$$

In this way we find the solution the stationary DN-diffraction problem in angle as the limiting amplitude of the corresponding nonstationary DN-diffraction problem. The obtained formula (2.12) is similar to the well-known Sommerfeld-Malyuzhinets from [10]–[13].

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Behavior of a Class of Second-order Planar Elliptic Equations with Degeneracies

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Abstract. Normalization results are obtained for classes of second-order elliptic equations in $\mathbb{R}^2$ which degenerate along a simple closed curve or with an isolated singularity. The behavior of the solutions of the corresponding homogeneous equation in a neighborhood of the degeneracy as well as the maximum principle is studied.

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1. Introduction

This paper deals with the properties of a class of degenerate elliptic partial differential equations in the plane. We consider the equations where the set of degeneracy is either a simple closed curve or an isolated singularity. More precisely, the equation dealt with here is the second-order homogeneous equation in $\mathbb{R}^2$ with smooth coefficients given by

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} = 0. \tag{1.1}$$

For the first class of equations, we assume that $AC - B^2 > 0$ except along a simple closed curve Σ. Along Σ, the equation is parabolic and we assume that the coefficients satisfy an additional condition on the order of degeneracy (see Section 2). The second class consists of equations for which $AC - B^2 > 0$ except at an isolated point 0. Near the singular point, we assume that

$$K_1(x^2 + y^2)^2 \leq AC - B^2 \leq K_2(x^2 + y^2)^2$$

for some positive constants $K_1 < K_2$.

We use the results obtained by the author in [5] for the operator

$$(x^2 + y^2)^2\Delta + p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}$$

to study the properties of the solutions of equation (1.1) in a neighborhood of the set where the equation degenerates. Some of the results obtained include the existence of Hölder continuous solutions and the validity of the maximum principle.

There is a vast amount of work dealing with elliptic equations with degeneracies. This paper is in the spirit of the works contained in [2] and in [7].

The organization of this paper is as follows. In Sections 2 and 3, we prove normalization results for equation (1.1). In Section 4, we recall needed results from [5]. The main results of this paper are proved in Section 5.

2. Normalization near a curve of degeneracy

In this section we prove a normalization for a class of second-order elliptic operators which degenerate along a simple closed curve. To achieve such normalization, we will make use of the following theorem proved in [3]

Theorem 2.1. *Let X be a C^∞ complex vector field in $\mathbb{R}^2$ satisfying the following conditions in a neighborhood of a smooth, simple, closed curve Σ:*

(i) $X_p \wedge \overline{X}_p \neq 0$ *for every* $p \notin \Sigma$*;*
(ii) $X_p \wedge \overline{X}_p$ *vanishes to first order for* $p \in \Sigma$*; and*
(iii) X *is tangent to* Σ*.*

Then there exist an open tubular neighborhood U of Σ, a positive number R, a unique complex number $\lambda \in \mathbb{R}^+ + i\mathbb{R}$, and a diffeomorphism

$$\Phi : U \longrightarrow (-R,\ R) \times \mathbb{S}^1$$

such that

$$\Phi_* X = m(r,t)\left[\lambda\frac{\partial}{\partial t} - ir\frac{\partial}{\partial r}\right]$$

where $m(r,t)$ is a nonvanishing function. Moreover, when $\lambda \notin \mathbb{Q}$, then for any given $k \in \mathbb{Z}^+$, the diffeomorphism Φ and the function m can be taken to be of class C^k.

This theorem was proved in [3] when $\mathrm{Im}(\lambda) \neq 0$. When $\mathrm{Im}(\lambda) = 0$, it is proved in [3] that the diffeomorphism Φ is C^1. Later it was proved in [1] that Φ can be taken to be of class C^k provided that $\lambda \notin \mathbb{Q}$.

Let $\mathbb{P}$ be the second-order operator in $\mathbb{R}^2$ given by

$$\mathbb{P} = A_{11}\frac{\partial^2}{\partial x^2} + 2A_{12}\frac{\partial^2}{\partial x \partial y} + A_{22}\frac{\partial^2}{\partial y^2} + A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y}\,, \qquad (2.1)$$

where the coefficients $A_{11}, \ldots, A_2$ are C^∞ functions. We assume that $\mathbb{P}$ is elliptic everywhere except along a C^∞ simple closed curve Σ. Thus,

$$\begin{aligned} A_{11}(x,y)A_{22}(x,y) - A_{12}^2(x,y) > 0 \quad \forall (x,y) \notin \Sigma; \\ A_{11}(x,y)A_{22}(x,y) - A_{12}^2(x,y) = 0 \quad \forall (x,y) \in \Sigma. \end{aligned} \tag{2.2}$$

We assume further that the degeneracy of $\mathbb{P}$ on Σ satisfies the following condition: For every $p \in \Sigma$, there are coordinates (u,v) centered at p such that:

(C_1) : The curve Σ is given near p by $u = 0$;

(C_2) : The expression of $\mathbb{P}$ in the (u,v) coordinates is a multiple of an operator

$$\frac{\partial^2}{\partial u^2} + 2uB_{12}\frac{\partial}{\partial u \partial v} + u^2 B_{22}\frac{\partial}{\partial v^2} + uB_1\frac{\partial}{\partial u} + B_2\frac{\partial}{\partial v},$$

where $B_{12}, \ldots$ are C^∞ functions.

We start by choosing a coordinate system in which Σ is a circle.

Proposition 2.2. *There exists a C^∞ diffeomorphism Ψ from a tubular neighborhood U of $\Sigma \subset \mathbb{R}^2$ onto a cylinder $(-\delta,\ \delta) \times \mathbb{S}^1$ such that the pushforward $\Psi_*\mathbb{P}$ is*

$$\Psi_*\mathbb{P} = m(\rho,\theta)\left[\frac{\partial^2}{\partial\theta^2} + 2\rho N\frac{\partial^2}{\partial\rho\partial\theta} + \rho^2 M\frac{\partial^2}{\partial\rho^2} + \rho Q\frac{\partial}{\partial\rho} + T\frac{\partial}{\partial\theta}\right] \tag{2.3}$$

with $m(\rho,\theta) \neq 0$ for every (ρ,θ) and where the functions M and N satisfy

$$M(\rho,\theta) - N^2(\rho,\theta) \geq K, \qquad \forall(\rho,\theta) \in (-\delta,\delta) \times \mathbb{S}^1 \tag{2.4}$$

for some positive constant K.

Proof. Since Σ is a C^∞ simple closed curve, then we can find a differeomorphism Ψ_1 that sends Σ onto the unit circle in $\mathbb{R}^2$. Consider the diffeomorphism Ψ_2 from the cylinder $(-\epsilon,\epsilon) \times \mathbb{S}^1$ onto the annulus $1-\epsilon < |(x,y)| < 1+\epsilon$ in $\mathbb{R}^2$, given by

$$\Psi_2(\rho,\theta) = ((\rho+1)\cos\theta, (\rho+1)\sin\theta).$$

If we take $\Psi = \Psi_2 \circ \Psi_1^{-1}$, then it follows at once from the ellipticity of $\mathbb{P}$ outside of Σ and from condition C_2, that $\Psi_*\mathbb{P}$ has the desired form given in the proposition $\square$

For the operator $\mathbb{P}$, we have the following normalization

Theorem 2.3. *Let $\mathbb{P}$ be the second-order operator given by* (1.1) *whose coefficients satisfy conditions C_1, C_2 along the degeneracy curve Σ. Then there is a unique number $\lambda \in \mathbb{R}^+ + i\mathbb{R}$ and a diffeomorphism Φ*

$$\Phi : U \longrightarrow (-\delta,\ \delta) \times \mathbb{S}^1,$$

where U is a tubular neighborhood of Σ, such that

$$\Phi_*\mathbb{P} = m(r,t)\left[L\overline{L} + \mathrm{Re}\,(\beta(r,t)L)\right] \tag{2.5}$$

where

$$L = \lambda\frac{\partial}{\partial t} - ir\frac{\partial}{\partial r}$$

and $\overline{L}$ is complex conjugate vector field of L, and where m and β are functions with m nowhere vanishing and β is $\mathbb{C}$-valued.

Proof. Thanks to Proposition 2.2, we can assume that

$$\mathbb{P} = \frac{\partial^2}{\partial\theta^2} + 2\rho N\frac{\partial^2}{\partial\rho\partial\theta} + \rho^2 M\frac{\partial^2}{\partial\rho^2} + \rho Q\frac{\partial}{\partial\rho} + T\frac{\partial}{\partial\theta}\,. \tag{2.6}$$

Let X be the vector field given by

$$X = \frac{\partial}{\partial\theta} - \rho g(\rho,\theta)\frac{\partial}{\partial\rho}$$

where the function g is given by

$$g(\rho,\theta) = N(\rho,\theta) + i\sqrt{M(\rho,\theta) - N^2(\rho,\theta)}.$$

We have

$$X\overline{X} = \frac{\partial^2}{\partial\theta^2} - 2\rho N\frac{\partial^2}{\partial\rho\partial\theta} + \rho^2 M\frac{\partial^2}{\partial\rho^2} - \rho f\frac{\partial}{\partial\rho} \tag{2.7}$$

with

$$f = \frac{X(\rho\overline{g})}{\rho} = \overline{g}_\theta - g(\rho\overline{g})_\rho = \overline{g}_\theta - |g|^2 - \rho g\overline{g}_\rho.$$

Note that since

$$\rho\frac{\partial}{\partial\rho} = \frac{X - \overline{X}}{\overline{g} - g}, \quad \text{and} \quad \frac{\partial}{\partial\theta} = \frac{\overline{g}X - g\overline{X}}{\overline{g} - g},$$

then it follows from (2.6), (2.7) and from the fact that the coefficients of $\mathbb{P}$ are $\mathbb{R}$-valued that

$$\begin{aligned}\mathbb{P} &= X\overline{X} + (f+Q)\frac{X-\overline{X}}{\overline{g}-g} + T\frac{\overline{g}X - g\overline{X}}{\overline{g}-g} = X\overline{X} + \frac{f+Q+\overline{g}T}{\overline{g}-g}X - \frac{f+Q+gT}{\overline{g}-g}\overline{X}\\ &= \overline{X}X + \frac{\overline{f}+Q+gT}{g-\overline{g}}\overline{X} - \frac{\overline{f}+Q+\overline{g}T}{g-\overline{g}}X\end{aligned}$$

Hence,

$$2\mathbb{P} = X\overline{X} + \overline{X}X + A(\rho,\theta)X + \overline{A(\rho,\theta)}\,\overline{X}\,, \tag{2.8}$$

with

$$A = \frac{f + \overline{f} + 2Q + 2\overline{g}T}{g - \overline{g}}\,.$$

The vector field X is elliptic outside the circle $\Sigma = \{0\} \times \mathbb{S}^1$; it is tangent to Σ; and

$$X \wedge \overline{X} = -\rho(g - \overline{g})\frac{\partial}{\partial\rho} \wedge \frac{\partial}{\partial\theta} = -2i\rho\sqrt{M - N^2}\frac{\partial}{\partial\rho} \wedge \frac{\partial}{\partial\theta}\,,$$

vanishes to first order along Σ. Hence, it follows from Theorem 2.1 that there exist $\lambda \in \mathbb{R}^+ + i\mathbb{R}$ and a diffeomorphism Φ defined in a tubular neighborhood of Σ onto a cylinder $(-\delta,\ \delta) \times \mathbb{S}^1$ such that $\Phi_* X = m_1 L$, where L is the vector defined in Theorem 2.2. We have then from (2.8) that

$$\Phi_*\mathbb{P} = |m_1|^2 \mathrm{Re}\left[L\overline{L} + \frac{m_1 A + \overline{m_1}\overline{L}m_1}{|m_1|^2}L\right]$$

This completes the proof. □

Remark 2.4. It follows from [3] that the invariant λ appearing in L is given by

$$\frac{1}{\lambda} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sqrt{M(0,\theta)-N(0,\theta)^2} - iN(0,\theta)\right)\,d\theta.$$

3. Normalization near an isolated singularity

In this section, we show that a normalization can be achieved for a class of second-order operators with isolated singularities.

Let $\mathbb{D}$ be the second-order operator given in a neighborhood of $0 \in \mathbb{R}^2$ by

$$\mathbb{D}u = a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y \tag{3.1}$$

where the coefficients $a_{11}, \ldots, a_2$ are $\mathbb{R}$-valued, of class C^∞, and vanishing at 0. We assume that a_{11} is nonnegative, and

$$C_1 \le \frac{a_{11}(x,y)a_{22}(x,y) - a_{12}(x,y)^2}{(x^2+y^2)^2} \le C_2 \tag{3.2}$$

for some positive constants $C_1 < C_2$. It follows in particular that a_{11} and a_{22} vanish to second order at 0. To the operator $\mathbb{D}$, we associate the functions A and B defined for $(x,y) \neq 0$ by

$$\begin{aligned} A(x,y) &= \frac{(x^2+y^2)\sqrt{a_{11}a_{22}-a_{12}^2}}{a_{11}y^2 - 2a_{12}xy + a_{22}x^2} \\ B(x,y) &= \frac{(a_{22}-a_{11})xy + a_{12}(x^2-y^2)}{a_{11}y^2 - 2a_{12}xy + a_{22}x^2} \end{aligned} \tag{3.3}$$

Note that it follows from (3.2) that these functions are bounded and A is positive. Let

$$\mu = \frac{1}{2\pi}\lim_{\rho\to 0^+}\int_{C_\rho}\frac{A(x,y)-iB(x,y)}{x^2+y^2}\,(xdy - ydx)\,, \tag{3.4}$$

where C_ρ denotes the circle with radius ρ and center 0. This number is introduced in [6]. We will prove that $\mu \in \mathbb{R}^+ + i\mathbb{R}$ is well defined and it is an invariant for the operator $\mathbb{D}$. In what follows, we will be using polar coordinates $x = \rho\cos\theta$, $y = \rho\sin\theta$ and we will denote this change of coordinates by Ψ. Thus,

$$\Psi : \mathbb{R}^2\backslash 0 \longrightarrow \mathbb{R}^+ \times \mathbb{S}^1\,, \qquad \Psi(x,y) = (\rho,\theta)\,.$$

The normalization of the operator $\mathbb{D}$ is given by the following theorem.

Theorem 3.1. *Let $\mathbb{D}$ be the second-order operator given by* (3.1) *whose coefficients vanish at* 0 *and satisfy condition* (3.2). *Then there exist a neighborhood U of the circle $\{0\}\times\mathbb{S}^1$ in $[0,\ \infty)\times\mathbb{S}^1$, a positive number R, a diffeomorphism*

$$\Phi : U \longrightarrow [0,\ R)\times\mathbb{S}^1$$

sending $\{0\}\times\mathbb{S}^1$ onto itself, such that

$$(\Phi\circ\Psi)_*\mathbb{D} = m(r,t)\left[L\overline{L} + \mathrm{Re}(\alpha(r,t)L)\right] \tag{3.5}$$

where m, α are differentiable functions with $m(r,t) \neq 0$, α is $\mathbb{C}$-valued and where

$$L = \lambda \frac{\partial}{\partial t} - ir\frac{\partial}{\partial r}\,, \quad \lambda = \frac{1}{\mu}$$

and μ is given by (3.4). Moreover, if the invariant $\mu \notin \mathbb{Q}$, then for every $k \in \mathbb{Z}^+$, the diffeomorphism Φ, and the functions m, and α can be chosen to be of class C^k.

Proof. We start by rewriting $\mathbb{D}$ in polar coordinates:

$$\mathbb{D}u = Pu_{\theta\theta} + 2Nu_{\rho\theta} + Mu_{\rho\rho} + Qu_\rho + Tu_\theta \tag{3.6}$$

where

$$\begin{aligned}
P &= \frac{1}{\rho^2}\left[a_{11}\sin^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta\right]\\
N &= \frac{1}{\rho}\left[-a_{11}\sin\theta\cos\theta + a_{12}(\cos^2\theta - \sin^2\theta) + a_{22}\cos\theta\sin\theta\right]\\
M &= a_{11}\cos^2\theta + 2a_{12}\sin\theta\cos\theta + a_{22}\sin^2\theta\\
Q &= \frac{1}{\rho}\left[a_{11}\sin^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta\right] + a_1\cos\theta + a_2\sin\theta\\
T &= \frac{1}{\rho^2}\left[a_{11}\sin\theta\cos\theta + a_{12}(\sin^2\theta - \cos^2\theta) - a_{22}\sin\theta\cos\theta\right]\\
&\qquad - \frac{1}{\rho}(a_1\sin\theta + a_2\cos\theta)
\end{aligned}$$

Condition (3.2) implies that there is a constant $C_0 >$ such that

$$M(\rho,\theta) \geq C_0\rho^2 \quad \text{and} \quad P(\rho,\theta) \geq C_0 \qquad \forall(\rho,\theta).$$

We define the following C^∞ functions (of (ρ,θ))

$$N_1 = \frac{N}{\rho P}, \quad M_1 = \frac{M}{\rho^2 P}, \quad Q_1 = \frac{Q}{\rho P}, \quad T_1 = \frac{T}{P}.$$

In terms of these functions, (3.2) takes the form

$$M_1(\rho,\theta) - N_1^2(\rho,\theta) \;\geq\; C_2, \qquad \forall(\rho,\theta) \in [0,\ R_1] \times \mathbb{S}^1\,, \tag{3.7}$$

and (3.6) becomes

$$\frac{\mathbb{D}u}{P} = u_{\theta\theta} + 2\rho N_1 u_{\rho\theta} + \rho^2 M_1 u_{\rho\rho} + \rho Q_1 u_\rho + T_1 u_\theta\,. \tag{3.8}$$

This expression is similar to that of $\mathbb{P}$ given in (2.6) and the proof continues as that of Theorem 2.3. It should be noted (see Remark 2.4) that the associated invariant λ is given by $\lambda = 1/\widetilde{\mu}$ where

$$\widetilde{\mu} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sqrt{M_1(0,\theta) - N_1^2(0,\theta)} - iN_1(0,\theta)\right)d\theta\,.$$

Note the $\widetilde{\mu} = \mu$ with μ given in (3.4). This can be seen by rewriting the integral appearing in (3.4) in polar coordinates. □

4. The Laplace equation with a singular point

The singular second-order equation

$$\Delta f + \frac{p(x,y)}{r} f_x + \frac{q(x,y)}{r} f_y = F(x,y)\,, \tag{4.1}$$

is studied in [5]. Here $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the Laplacian in $\mathbb{R}^2$, $r = \sqrt{x^2+y^2}$, p, q are Hölder continuous functions. The behavior of the solutions near the singular point 0 are well understood. Note that equation (4.1) can be rewritten as

$$\frac{\partial^2 f}{\partial z \partial \overline{z}} - 2\mathrm{Re}\left(\frac{\mu(z)}{r}\frac{\partial f}{\partial z}\right) = \frac{F(z)}{4}\,, \tag{4.2}$$

where we have set $z = x + iy = r\mathrm{e}^{i\theta}$,

$$\mu(z) = \frac{p(z) + iq(z)}{4}, \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

In [5], equation (4.1) or equivalently (4.2) is studied when the coefficients p and q are Hölder continuous in a neighborhood of 0. In fact the results of [5] can be generalized to the case when p and q (or equivalently μ) satisfy weaker conditions. In particular, they can be generalized to the case when the coefficients are bounded in a neighborhood of 0 and are given in polar coordinates as

$$p(r,\theta) = p_0(\theta) + r^{\nu} p_1(r,\theta) \quad q(r,\theta) = q_0(\theta) + r^{\nu} q_1(r,\theta)\,,$$

where $\nu > 0$, $p_1(r,\theta)$ and $q_1(r,\theta)$ are continuous functions, $p_0(\theta)$ and $q_0(\theta)$ are continuous, 2π-periodic, and satisfy

$$\int_0^{2\pi} \mu(0,\theta)\mathrm{e}^{-i\theta}\, d\theta = \int_0^{2\pi} (p_0(\theta) + iq_0(\theta))\mathrm{e}^{-i\theta}\, d\theta = 0\,. \tag{4.3}$$

This generalization can be achieved as follows. In [5] equation (4.1) is studied by reducing it into a generalized CR equation with a singular point considered in [4]. This reduction relies on the function $A_0(\theta)$ defined in Lemma 3.1 of [5]. For more general coefficients satisfying (4.3), we can replace A_0 by the function

$$\widetilde{A}_0(\theta) = \exp\left(2\int_0^{\theta} (q_0(s) - ip_0(s))\mathrm{e}^{-is}\, ds\right).$$

With this adjustment of the function A_0, the results proved when p, q are continuous at 0 carry over easily (almost verbatim) to the case when p, q satisfy (4.3). In particular, we have the following results for the homogeneous equation

$$\frac{\partial^2 f}{\partial z \partial \overline{z}} - 2\mathrm{Re}\left(\frac{\mu(z)}{r}\frac{\partial f}{\partial z}\right) = 0\,. \tag{4.4}$$

Theorem 4.1. *There exists a sequence of real numbers*

$$\cdots \sigma_{-1}^- \le \sigma_{-1}^+ < \sigma_0^- \le \sigma_0^+ < \sigma_1^- \le \sigma_1^+ < \cdots$$

with

$$\lim_{j\to-\infty}\sigma_j^{\pm}=-\infty \quad\text{and}\quad \lim_{j\to\infty}\sigma_j^{\pm}=\infty$$

such that if f is of finite order at 0 and solves equation (4.4) *in a neighborhood of 0, then there exist $j_0\in\mathbb{Z}$ and $\tau=\sigma_{j_0}^-$ or $\tau=\sigma_{j_0}^+$ such that f has order τ at 0. Thus*

$$K_1(x^2+y^2)^{\tau/2}\le |f(x,y)|\le K_2(x^2+y^2)^{\tau/2}\quad \forall (x,y)\neq 0.$$

Furthermore, the index j_0 is the winding number about 0 of the function f_z, where $2f_z=f_x-if_y$.

In the statement of the Theorem 4.1, by a function f of finite order at 0, we mean that there exists $N>0$ such $(x^2+y^2)^N f$ is bounded near 0.

Remark 4.2. The sequence $\{\sigma_j^{\pm}\}$ represents the spectral values of an associated 2×2 system of first-order ordinary differential equations with periodic coefficients. For a given spectral value, the index $j\in\mathbb{Z}$ denotes the winding number of the corresponding periodic solutions of the system. When the system has two independent periodic solutions, then $\sigma_j^-=\sigma_j^+$, otherwise (when it has only one periodic solution), $\sigma_j^-<\sigma_j^+$.

Theorem 4.3. *If f is a continuous solution of* (4.4)*, then f is Hölder continuous.*

Theorem 4.4. *If $-1\notin\{\sigma_j^{\pm}\}_{j\in\mathbb{Z}}$, then any bounded solution of* (4.4) *is Hölder continuous.*

We will say that a function g defined in a neighborhood of $0\in\mathbb{R}^2$ is flat at 0 if

$$\lim_{(x,y)\to 0}\frac{g(x,y)-g(0)}{(x^2+y^2)^m}=0,\qquad \forall m\ge 0.$$

We have the following uniqueness result.

Theorem 4.5. *If f is a solution of* (4.4) *and if f is flat at 0, then f is constant.*

When the coefficients are C^∞, equation (4.4) has nontrivial regular solution. More precisely,

Theorem 4.6. *If the coefficients p, q of equation* (4.4) *are of class C^∞, then for any $k\in\mathbb{Z}^+$, equation* (4.4) *has nontrivial solutions that are of class C^k in a neighborhood of 0.*

The maximum principle also holds for equation (4.4)

Theorem 4.7. *Let f be a nonconstant Lipschitz solution of* (4.4)*, then $f(0)$ is not an extreme value of f. Consequently, if f is a Lipschitz solution of* (4.4) *on the closure $\overline{U}$ of an open set U containing 0 in its interior, then the maximum and minimum values of f occur on the boundary ∂U.*

In fact, the Lipschitz condition can be weakened when the spectral value σ_{-1}^+ satisfies $\sigma_{-1}^+\le -1$. More precisely,

Theorem 4.8. *Suppose that $\sigma_{-1}^+\le -1$. Let f be a nonconstant continuous solution of* (4.4)*, then $f(0)$ is not an extreme value of f.*

5. Properties of the equation $\mathbb{P}u = 0$ and $\mathbb{D}u = 0$

Many of the properties of the solutions of equation (4.1) involving Δ as the principal part extend to the solutions of the more general equations $\mathbb{P}u = 0$ and $\mathbb{D}u = 0$. Before we proceed further, we can assume, by using Theorem 2.1, that

$$\mathbb{P} = L\overline{L} + \mathrm{Re}\left(\beta(r,\theta)\overline{L}\right) \tag{5.1}$$

where

$$L = \lambda\frac{\partial}{\partial t} - ir\frac{\partial}{\partial r}\,.$$

We assume throughout that the invariant λ is real, so $\lambda \in \mathbb{R}^+$, and that $\beta(r,t) = \beta_0(t) + r^a\beta_1(r,t)$ with β_1 of class C^k ($k \geq 1$) and β_0 satisfying the condition

$$\int_0^{2\pi} \beta_0(t)dt = 0 \tag{5.2}$$

We have the following result

Theorem 5.1. *There exists a sequence of real numbers*

$$\cdots < \sigma_{-1}^- \leq \sigma_{-1}^+ < \sigma_0^- \leq \sigma_0^+ < \sigma_1^- \leq \sigma_1^+ < \cdots$$

with

$$\lim_{j\to-\infty} \sigma_j^\pm = -\infty\,, \quad \text{and} \quad \lim_{j\to\infty} \sigma_j^\pm = \infty$$

such that whenever $u(r,t)$ solves the equation $\mathbb{P}u = 0$ in a cylinder $(0,\ \delta) \times \mathbb{S}^1$ (or in the cylinder $(-\delta,\ 0) \times \mathbb{S}^1$) and $|r|^s u(r,t)$ is bounded function for some $s \in \mathbb{R}$, then there exists a $j_0 \in \mathbb{Z}$ and $\tau = \sigma_{j_0}^+$ or $\tau = \sigma_{j_0}^-$ such that

$$K_1|r|^{\lambda\tau} \leq |u(r,t)| \leq K_2|r|^{\lambda\tau}, \qquad \forall(r,t),\ 0 < |r| < \delta \tag{5.3}$$

Proof. For $r > 0$, let $Z(r,t) = r^\lambda \mathrm{e}^{it}$. Then Z is a first integral of the vector field L in the cylinder $r > 0$. That is, $LZ = 0$ and $dZ \neq 0$. Furthermore,

$$Z : \mathbb{R}^+ \times \mathbb{S}^1 \longrightarrow \mathbb{C}^*$$

is a diffeomorphism. The pushforward of L via the map Z is the singular CR operator:

$$Z_*L = -2i\lambda\overline{Z}\,\frac{\partial}{\partial\overline{Z}}.$$

Hence,

$$Z_*\mathbb{P} = 4\lambda|Z|^2\left[\frac{\partial^2}{\partial Z\partial\overline{Z}} + \mathrm{Re}\left(\frac{\mu(|Z|,t)}{|Z|}\,\frac{\partial}{\partial Z}\right)\right] \tag{5.4}$$

where

$$\mu(|Z|,t) = \frac{i}{2\lambda}\beta(|Z|^{1/\lambda},t)\mathrm{e}^{it} \tag{5.5}$$

Note that since β satisfies (5.2), then μ satisfies condition (4.3). Let $\{\sigma_j^\pm\}_{j\in\mathbb{Z}}$ be the spectrum of the equation

$$\frac{\partial^2 v}{\partial Z\partial\overline{Z}} + \mathrm{Re}\left(\frac{\mu(|Z|,t)}{|Z|}\,\frac{\partial v}{\partial Z}\right) = 0 \tag{5.6}$$

Suppose that the function $u(r,t)$ solves the equation $\mathbb{P}u = 0$ in a cylinder $0 < r < \delta$ and that $r^s u$ is bounded for some $s \in \mathbb{R}$. The function $v(Z)$ defined in polar coordinates $Z = \rho e^{it}$ by

$$v(\rho, t) = u(\rho^{1/\lambda}, t) \tag{5.7}$$

is of finite order at $0 \in \mathbb{C}$ and satisfies the equation (5.6). It follows from Theorem 4.1 that there exists $j_0 \in \mathbb{Z}$ and $\tau = \sigma_{j_0}^{\pm}$ such that

$$K_1|Z|^\tau \le |v(Z)| \le K_2|Z|^\tau$$

for some positive constants $K_1 < K_2$. This means that the function $u(r,t) = v(r^\lambda, t)$ satisfies (5.3).

The case $r < 0$ can be handled in a similar way by considering the first integral of L given by $Z = (-r)^\lambda e^{-it}$. □

The reduction of the equation $\mathbb{P}u = 0$ into the singular Laplace equation (4.6) allows to extend the rest of the theorems in Section 4. More precisely, we have

Theorem 5.2. *If u is a continuous solution of $\mathbb{P}u = 0$ in a cylinder $(-\delta,\ \delta) \times \mathbb{S}^1$, then u is Hölder continuous and u is constant along the circle $\{0\} \times \mathbb{S}^1$.*

The constancy of u along the characteristic circle follows from the continuity at 0 of the function v defined in the proof of Theorem 5.1.

We will say that a function $g(r,t)$ defined in a cylinder $(-\delta,\ \delta) \times \mathbb{S}^1$ is flat on the circle $r = 0$ if

$$\lim_{r\to 0} \frac{g(r,t) - g(0,t)}{r^m} = 0, \qquad \forall m \ge 0.$$

We have the following uniqueness result.

Theorem 5.3. *If u is a solution of $\mathbb{P}u = 0$ and if u is flat on the circle $r = 0$, then u is constant.*

When the coefficient $\beta(r,t)$ is C^∞, equation $\mathbb{P}u = 0$ has nontrivial regular solution. More precisely,

Theorem 5.4. *If the coefficient $\beta \in C^\infty((-\delta,\ \delta) \times \mathbb{S}^1)$, then for any $k \in \mathbb{Z}^+$, equation $\mathbb{P}u = 0$ has nontrivial solutions that are of class C^k in a neighborhood of the circle $r = 0$.*

For the maximum principle, we need to assume Höder continuity along $r = 0$ with exponent $\alpha \ge 1/\lambda$. This guarantees that the function v given in (5.7) is Lipschitz at $0 \in \mathbb{R}^2$. We have then the following theorem.

Theorem 5.5. *Let u be a nonconstant solution of $\mathbb{P}u = 0$ in a an open set U containing the circle $r = 0$. Assume that u is Hölder continuous on $r = 0$ with an exponent $\ge 1/\lambda$. Then u cannot achieve an extreme values at any point on the circle $r = 0$. Thus, if in addition u is continuous on the bounded domain $\overline{U}$, then the maximum and minimum values of u occur on the boundary ∂U.*

For the equation $\mathbb{D}u = 0$, where $\mathbb{D}$ is the operator given in Section 3 whose coefficients satisfy condition (3.2), Theorem 3.1 allows us to consider the equation $\mathbb{D}u = 0$ as an equation $\mathbb{P}u = 0$, in an appropriate system of coordinates outside the singular point. The results proved here for $\mathbb{P}$ have therefore their counterpart for the operator $\mathbb{D}$. In particular, the order of a solution u is well defined through the associated spectral values, and the maximum principle also holds for the operator $\mathbb{D}$.

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Finite Sections of Band-dominated Operators on Discrete Groups

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Abstract. Let Γ be a finitely generated exact discrete group. We consider operators on $l^2(\Gamma)$ which are composed by operators of multiplication by a function in $l^\infty(\Gamma)$ and by the operators of left-shift by elements of Γ. These operators generate a C^*-subalgebra of $L(l^2(\Gamma))$ the elements of which we call band-dominated operators on Γ. We study the stability of the finite sections method for band-dominated operators with respect to a given generating system of Γ. Our approach is based on the equivalence of the stability of a sequence and the Fredholmness of an associated operator, and on Roe's criterion for the Fredholmness of a band-dominated operator on an exact discrete group, which we formulate in terms of limit operators. Special emphasis is paid to the quasicommutator ideal of the algebra generated by the finite sections sequences and to the stability of sequences in that algebra. For both problems, the sequence of the discrete boundaries plays an essential role.

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1. Introduction

Let Γ be a countable (not necessarily commutative) discrete group. We write the group operation as multiplication and let e stand for the identity element of Γ. For each non-empty subset X of Γ, let $l^2(X)$ stand for the Hilbert space of all functions $f : X \to \mathbb{C}$ with

$$\|f\|^2 := \sum_{x\in X} |f(x)|^2 < \infty.$$

For $X = \emptyset$, we define $l^2(X)$ as the space $\{0\}$ consisting of the zero element only. We consider $l^2(X)$ as a closed subspace of $l^2(\Gamma)$ in a natural way. The orthogonal

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projection from $l^2(\Gamma)$ to $l^2(X)$ will be denoted by P_X. Thus, P_Γ and $P_\emptyset$ are the identity and the zero operator, respectively. For $s \in \Gamma$, let δ_s be the function on Γ which is 1 at s and 0 at all other points. The family $(\delta_s)_{s\in\Gamma}$ forms an orthonormal basis of $l^2(\Gamma)$, to which we refer as the standard basis.

The *left regular representation* $L : \Gamma \to L(l^2(\Gamma))$ of Γ associates with every group element r a unitary operator L_r such that $L_r\delta_s = \delta_{rs}$ for $s \in \Gamma$. Since $\delta_{rs}(t) = \delta_s(r^{-1}t)$, one has $(L_r u)(t) = u(r^{-1}t)$ for every $u \in l^2(\Gamma)$. Hence, $r \mapsto L_r$ is a group isomorphism. Further, we associate with each function $a \in l^\infty(\Gamma)$ the operator aI of multiplication by a, i.e., $(au)(t) = a(t)u(t)$ for $u \in l^2(\Gamma)$. The smallest closed subalgebra of $L(l^2(\Gamma))$ which contains all operators L_r with $r \in \Gamma$ and aI with $a \in l^\infty(\Gamma)$ is called the *algebra of the band-dominated operators on* Γ. We denote it by $\mathsf{BDO}(\Gamma)$. Besides $\mathsf{BDO}(\Gamma)$ we consider the smallest closed subalgebra $\mathsf{Sh}(\Gamma)$ of $L(l^2(\Gamma))$ which contains all "shift" operators L_r with $r \in \Gamma$. Clearly, the algebras $\mathsf{BDO}(\Gamma)$ and $\mathsf{Sh}(\Gamma)$ are symmetric and, hence, C^*-subalgebras of $L(l^2(\Gamma))$.

Let $\mathcal{Y} = (Y_n)_{n=1}^\infty$ be an increasing sequence of finite subsets of Γ with $\cup_{n\geq 1} Y_n = \Gamma$. A sequence $(A_n)_{n=1}^\infty$ of operators $A_n : \operatorname{im} P_{Y_n} \to \operatorname{im} P_{Y_n}$ is called *stable* if there is an $n_0 \geq 1$ such that the operators A_n are invertible for $n \geq n_0$ and the norms of their inverses A_n^{-1} are bounded uniformly with respect to $n \geq n_0$. Note that stability is crucial for many questions in asymptotic numerical analysis. It dominates topics like the approximate solution of operator equations and the approximate spectral and pseudo-spectral theory. For a detailed overview see [5].

Let $A \in L(l^2(\Gamma))$. The operators $P_{Y_n} A P_{Y_n} : \operatorname{im} P_{Y_n} \to \operatorname{im} P_{Y_n}$ are called the *finite sections* of A with respect to $\mathcal{Y}$. In this paper, we are interested in the stability of the finite sections sequence $(P_{Y_n} A P_{Y_n})$ when $A \in \mathsf{BDO}(\Gamma)$. The finite sections method for band-dominated operators on the group $\mathbb{Z}$ of the integers is quite well understood, see [10, 11, 12, 13]. Finite sections for operators in $\mathsf{Sh}(\Gamma)$ with an arbitrary exact countable discrete group Γ were considered in [15].

Our approach to study the stability of the finite sections method for operators in $\mathsf{BDO}(\Gamma)$ is close to that in [13, 15]. We make use of the fact that a sequence (A_n) is stable if and only if an associated operator has the Fredholm property. In case the A_n are the finite sections of a band-dominated operator, the associated operator is a band-dominated operator again. So the desired stability result will finally follow from Roe's criterion for the Fredholm property of band-dominated operators in [17]. We thus start with recalling Roe's result in Section 2.

In Section 3, we provide an algebraic frame to study the stability of operator sequences. We introduce the C^*-algebra $\mathcal{S}_\mathcal{Y}(\mathsf{BDO}(\Gamma))$ generated by all finite sections sequences $(P_{Y_n} A P_{Y_n})$ with $A \in \mathsf{BDO}(\Gamma)$ and show that this algebra splits into the direct sum of $\mathsf{BDO}(\Gamma)$ and of an ideal which can be characterized as the quasicommutator ideal of the algebra. A main result is that the sequence $(P_{\partial Y_n})$ of the discrete boundaries always belongs to the algebra $\mathcal{S}_\mathcal{Y}(\mathsf{BDO}(\Gamma))$, and that this sequence already generates the quasicommutator ideal. This surprising fact has been already observed in other settings, for example for the algebras $\mathcal{S}(\mathsf{T}(C))$ of the finite sections method for the Toeplitz operators (a classical result, closely related to the present paper) and $\mathcal{S}_\mathcal{Y}(\mathsf{Sh}(\Gamma))$ (see [6] for the group $\Gamma = \mathbb{Z}^n$ and [15] for the general case), but also for the finite sections algebra $\mathcal{S}(\mathsf{O}_N)$ related with a concrete representation O_N of the Cuntz algebra (see [14]).

The final Section 4 is devoted to the prove of the stability theorem. We employ Roe's criterion using the limit operators language from [11]. The main task is to compute all (or at least a sufficient number of) limit operators of the band-dominated operator associated with a finite sections sequence.

2. The algebra of the band-dominated operators

We start with some alternate characterizations of band-dominated operators and the algebra generated by them. Consider functions $k \in l^\infty(\Gamma \times \Gamma)$ with the property that there is a finite subset Γ_0 of Γ such that $k(t, s) = 0$ whenever $ts^{-1} \notin \Gamma_0$. Then

$$(Au)(t) := \sum_{s \in \Gamma} k(t, s)\, u(s), \qquad t \in \Gamma, \tag{2.1}$$

defines a linear operator A on the linear space of all functions $u : \Gamma \to \mathbb{C}$, since the occurring series is finite for every $t \in G$. We call operators of this form *band operators* and the set Γ_0 a *band-width* of A.

Proposition 2.1. *An operator in $L(l^2(\Gamma))$ is a band operator if and only if it can be written as a finite sum $\sum b_i L_{t_i}$ where $b_i \in l^\infty(\Gamma)$ and $t_i \in \Gamma$.*

Proof. Let A be an operator of the form (2.1) and let $\Gamma_0 := \{t_1, t_2, \ldots, t_r\} \subseteq \Gamma$ be a finite set such that $k(t, s) = 0$ if $ts^{-1} \notin \Gamma_0$. Then,

$$(Au)(t) = \sum_{i=1}^{r} k(t, t_i^{-1}t)\, u(t_i^{-1}t) \quad \text{for } t \in \Gamma.$$

Set $b_i(t) := k(t, t_i^{-1}t)$. The functions b_i are in $l^\infty(\Gamma)$, and

$$A = \sum_{i=1}^{r} b_i L_{t_i}. \tag{2.2}$$

Conversely, each operator L_t with $t \in \Gamma$ is a band operator with band width $\{t\}$, and each operator bI with $b \in l^\infty(\Gamma)$ is a band operator with band width $\{e\}$. Since the band operators form an algebra, each finite sum $\sum b_i L_{t_i}$ is a band operator. $\square$

It is easy to see that the representation of a band operator on Γ in the form (2.2) with $b_i \neq 0$ is unique. The functions b_i are called the *diagonals* of the operator A. In particular, operators in $\mathsf{Sh}(\Gamma)$ can be considered as band-dominated operators with constant coefficients.

It is easy to see that the band operators form a symmetric algebra of bounded operators on $l^2(\Gamma)$. The norm closure of that algebra is just the algebra $\mathsf{BDO}(\Gamma)$, and this is why we call the elements of that algebra *band-dominated* operators.

The algebras $\mathsf{BDO}(\Gamma)$ and $\mathsf{Sh}(\Gamma)$ occur at many places and under different names in the literature. The algebra $\mathsf{Sh}(\Gamma)$ is *-isomorphic to the *reduced group*

C^*-algebra $C_r^*(\Gamma)$ in a natural way (see Section 2.5 in [3]). It can thus be considered as a concrete faithful representation of $C_r^*(\Gamma)$. Note also that the reduced group C^*-algebra coincides with the universal group C^*-algebra $C^*(\Gamma)$ if the group Γ is amenable. For this and further characterizations of amenable groups, see Theorem 2.6.8 in [3]. The algebra $\mathsf{BDO}(\Gamma)$ occurs in coarse geometry and is known there as the *uniform Roe algebra* or the *reduced translation algebra* ([16]). It can be identified with the reduced crossed product of the C^*-algebra $l^\infty(\Gamma)$ with the group Γ when the group action $\alpha : \Gamma \to \operatorname{Aut} l^\infty(\Gamma)$ is specified as

$$(\alpha_g f)(t) := f(g^{-1}t)$$

for $f \in l^\infty(\Gamma)$ and $g, t \in \Gamma$. Note that amenability of Γ is not needed for the following result. But if Γ is amenable, then the reduced crossed product $l^\infty(\Gamma) \times_{\alpha r} \Gamma$ coincides with the full crossed product $l^\infty(\Gamma) \times_\alpha \Gamma$ (see [7], Theorem 7.7.7 and [4], Corollary VII.2.2).

Theorem 2.2. *The reduced crossed product $l^\infty(\Gamma) \times_{\alpha r} \Gamma$ of the C^*-dynamical system $(l^\infty(\Gamma), \Gamma, \alpha)$ is *-isomorphic to* $\mathsf{BDO}(\Gamma)$.

Proof. Let $l^2(\Gamma, l^2(\Gamma))$ stand for the Hilbert space of all functions $x : \Gamma \to l^2(\Gamma)$ with $\sum_{s \in \Gamma} \|x(s)\|^2 < \infty$. For $a \in l^\infty(\Gamma)$, let $\pi(a)$ denote the operator aI of multiplication by a on $l^2(\Gamma)$ and define an operator $\tilde{\pi}(a)$ on $l^2(\Gamma, l^2(\Gamma))$ by

$$(\tilde{\pi}(a)x)(s) := \pi(\alpha_s^{-1}(a))(x(s)).$$

For $g \in \Gamma$, let $\tilde{L}_g$ be the operator on $l^2(\Gamma, l^2(\Gamma))$ defined by

$$(\tilde{L}_g x)(s) := x(t^{-1}s).$$

The pair $(\tilde{\pi}, \tilde{L})$ constitutes a covariant representation of the C^*-dynamical system $(l^\infty(\Gamma), \Gamma, \alpha)$ on $l^2(\Gamma, l^2(\Gamma))$. By the definition of the reduced crossed product (see [2, 4, 7], for instance), $l^\infty(\Gamma) \times_{\alpha r} \Gamma$ is the smallest C^*-subalgebra of $L(l^2(\Gamma, l^2(\Gamma)))$ which contains all operators $\tilde{\pi}(a)$ and $\tilde{L}_g$ with $a \in l^\infty(\Gamma)$ and $g \in \Gamma$. One can show ([7], Theorem 7.7.5) that each faithful representation (π', H) of $l^\infty(\Gamma)$ in place of the representation $(\pi, l^2(\Gamma))$ leads to the same algebra.

We identify $l^2(\Gamma, l^2(\Gamma))$ with $l^2(\Gamma \times \Gamma)$ via the mappings

$$J : l^2(\Gamma, l^2(\Gamma)) \to l^2(\Gamma \times \Gamma), \ (Jx)(s, n) := (x(s))(n),$$

$$J^{-1} : l^2(\Gamma \times \Gamma) \to l^2(\Gamma, l^2(\Gamma)), \ ((J^{-1}y)(s))(n) := y(s, n)$$

and determine the corresponding operators

$$\hat{\pi}(a) := J\tilde{\pi}(a)J^{-1} \quad \text{and} \quad \hat{L}_g := J\tilde{L}_g J^{-1}.$$

A straightforward calculation gives one has

$$(\hat{\pi}(a)x)(s, n) = a(sn)x(s, n) \quad \text{and} \quad (\hat{L}_g x)(s, n) = x(g^{-1}s, n). \tag{2.3}$$

Let $\mathcal{C}$ refer to the smallest C^*-subalgebra of $L(l^2(\Gamma \times \Gamma))$ which contains all operators $\hat{\pi}(a)$ and $\hat{L}_g$ with $a \in l^\infty(\Gamma)$ and $g \in \Gamma$, given by (2.3). For $n \in \Gamma$, let

$$H_n := \{x \in l^2(\Gamma \times \Gamma) : x(s, m) = 0 \text{ whenever } m \neq n\}.$$

We identify $l^2(\Gamma \times \Gamma)$ with the orthogonal sum $\oplus_{n\in\Gamma} H_n$ such that $x \in l^2(\Gamma \times \Gamma)$ is identified with $\oplus h_n \in \oplus H_n$ where $h_n(s) = x(s, n)$. From (2.3) we conclude that each space H_n is invariant with respect to each operator in $\mathcal{C}$ (i.e., $AH_n \subseteq H_n$ for $A \in \mathcal{C}$). Hence, each operator $A \in \mathcal{C}$ corresponds to a diagonal matrix operator $\mathrm{diag}\,(\ldots, A_n, A_{n+1}, \ldots)$ with respect to the decomposition of $l^2(\Gamma \times \Gamma)$ into the orthogonal sum of its subspaces H_n. Thus, A_n is the restriction of A onto H_n.

Let $\mathcal{C}_n$ be the C^*-algebra of all restrictions of operators in $\mathcal{C}$ onto H_n. It is clear that each of the spaces H_n is isometric to $l^2(\Gamma)$, with the isometry given by

$$J_n : H_n \to l^2(\Gamma), \qquad (J_n x)(s) := x(s, n),$$
$$J_n^{-1} : l^2(\Gamma) \to H_n, \quad (J_n^{-1} x)(s, n) := x(s).$$

Then

$$(J_n \hat{\pi}(a) J_n^{-1} x)(s) = (\hat{\pi}(a) J_n^{-1} x)(s, n) = (a(sn)(J^{-1}x))(s, n)$$
$$= a(sn)x(s) \ = \ (R_n \pi(a) R_n^{-1} x)(s)$$

where $(R_n f)(s) = f(sn)$ stands for the operator of the right-regular representation of Γ. Similarly,

$$(J_n \hat{L}_g J_n^{-1} x)(s) = (\hat{L}_g J_n^{-1} x)(s, n) = (J^{-1}x)(g^{-1}s, n)$$
$$= x(g^{-1}s) \ = \ (L_g x)(s).$$

Thus,

$$J_n \hat{\pi}(a) J_n^{-1} = R_n \pi(a) R_n^{-1} \quad \text{and} \quad J_n \hat{L}_g J_n^{-1} = L_g = R_n L_g R_n^{-1}.$$

Consequently, the mapping

$$\mathsf{BDO}(\Gamma) \to \mathcal{C}, \quad A \mapsto \mathrm{diag}\,(\ldots, J_n^{-1} R_n A R_n^{-1} J_n, \ldots)$$

is a *-isomorphism. Since $\mathcal{C}$ is evidently *-isomorphic to the reduced crossed product $l^\infty(\Gamma) \times_{\alpha r} \Gamma$, the assertion follows. □

Our next goal is to recall Roe's criterion [17] for the Fredholm property of band-dominated operators on $l^2(\Gamma)$. We are going to formulate this criterion in the language of limit operators.

Let $h : \mathbb{N} \to \Gamma$ be a sequence tending to infinity in the sense that for each finite subset Γ_0 of Γ, there is an $n_0 \in \mathbb{N}$ such that $h(n) \not\in \Gamma_0$ if $n \geq n_0$. Clearly, if h tends to infinity, then the inverse sequence h^{-1} tends to infinity, too. We say that an operator $A_h \in L(l^2(\Gamma))$ is a *limit operator of* $A \in L(l^2(\Gamma))$ *defined by the sequence* h if

$$R_{h(m)}^{-1} A R_{h(m)} \to A_h \quad \text{and} \quad R_{h(m)}^{-1} A^* R_{h(m)} \to A_h^*$$

strongly as $m \to \infty$ (as before, the R_r are given by the right-regular representation of Γ on $l^2(\Gamma)$). Clearly, every operator has at most one limit operator with respect to a given sequence h. Note that the generating function of the shifted operator $R_r^{-1} A R_r$ is related with the generating function of A by

$$k_{R_r^{-1} A R_r}(t, s) = k_A(tr^{-1}, sr^{-1}) \tag{2.4}$$

and that the generating functions of $R^{-1}_{h(m)}AR_{h(m)}$ converge pointwise on $\Gamma \times \Gamma$ to the generating function of the limit operator A_h (if the latter exists).

It is an important property of band-dominated operators that they always possess limit operators. More general, the following result can be proved by a standard Cantor diagonal argument (see [9, 10, 11]).

Proposition 2.3. *Let A be a band-dominated operator on $l^2(\Gamma)$. Then every sequence $h : \mathbb{N} \to \Gamma$ which tends to infinity possesses a subsequence g such that the limit operator A_g of A with respect to g exists.*

Let A be a band-dominated operator and $h : \mathbb{N} \to \Gamma$ a sequence tending to infinity for which the limit operator A_h of A exists. Let B be another band-dominated operator. By Proposition 2.3 we can choose a subsequence g of h such that the limit operator B_g exists. Then the limit operators of A, $A+B$ and AB with respect to g exist, and

$$A_g = A_h, \qquad (A+B)_g = A_g + B_g, \qquad (AB)_g = A_g B_g.$$

Thus, the mapping $A \mapsto A_h$ acts, at least partially, as an algebra homomorphism.

The following theorem is due to Roe [17], see also [8]. Recall that a group Γ is called *exact*, if its reduced translation algebra is exact as a C^*-algebra. The latter algebra is defined as the reduced crossed product of $l^\infty(\Gamma)$ by Γ and coincides with the C^*-algebra of all band-dominated operators on $l^2(\Gamma)$ in our setting. The class of exact groups is extremely rich. It includes all amenable groups (hence, all solvable groups such as the discrete Heisenberg group and the commutative groups) and all hyperbolic groups (in particular, all free groups with finitely many generators) (see [16], Chapter 3).

Theorem 2.4 (Roe). *Let Γ be a finitely generated discrete and exact group, and let A be a band-dominated operator on $l^2(\Gamma)$. Then the operator A is Fredholm on $l^2(\Gamma)$ if and only if all limit operators of A are invertible and if the norms of their inverses are uniformly bounded.*

Note that this result holds as well if the left regular representation is replaced by the right regular one and if, thus, the operators L_s and R_t change their roles. In fact, the results of [8, 17] are presented in this symmetric setting. In [8] we showed moreover that the uniform boundedness condition in Theorem 2.4 is redundant for band operators if the group Γ has sub-exponential growth and if not every element of Γ is cyclic in the sense that $w^n = e$ for some positive integer n. For details see [8]. Note that the condition of sub-exponential growth is satisfied by the abelian groups $\mathbb{Z}^N$, the discrete Heisenberg group and, more general, by nilpotent groups (in fact, these groups have polynomial growth), whereas the growth of the free group $\mathbb{F}_N$ with $N > 1$ is exponential.

Theorem 2.5. *Let Γ be a finitely generated discrete and exact group with sub-exponential growth which possesses at least one non-cyclic element, and let A be a band operator on $l^2(\Gamma)$. Then the operator A is Fredholm on $l^2(\Gamma)$ if and only if all limit operators of A are invertible.*

3. The algebra of the finite sections method

Given an increasing sequence $\mathcal{Y} := (Y_n)_{n\geq 1}$ of finite subsets of Γ such that $\cup_{n\geq 1} Y_n = \Gamma$, let $\mathcal{F}_\mathcal{Y}$ denote the set of all bounded sequences $\mathbf{A} = (A_n)$ of operators $A_n : \operatorname{im} P_{Y_n} \to \operatorname{im} P_{Y_n}$. Equipped with the operations

$$(A_n) + (B_n) := (A_n + B_n), \quad (A_n)(B_n) := (A_n B_n), \quad (A_n)^* := (A_n^*)$$

and the norm

$$\|\mathbf{A}\|_{\mathcal{F}_\mathcal{Y}} := \|A_n\|,$$

the set $\mathcal{F}_\mathcal{Y}$ becomes a C^*-algebra with identity $\mathbf{I} = (Y_n)$, and the set $\mathcal{G}_\mathcal{Y}$ of all sequences $(A_n) \in \mathcal{F}_\mathcal{Y}$ with $\lim \|A_n\| = 0$ forms a closed ideal of $\mathcal{F}_\mathcal{Y}$. The relevance of the algebra $\mathcal{F}_\mathcal{Y}$ and its ideal $\mathcal{G}_\mathcal{Y}$ in our context stems from the fact (following by a simple Neumann series argument) that a sequence $\mathbf{A} \in \mathcal{F}_\mathcal{Y}$ is stable if, and only if, its coset $\mathbf{A} + \mathcal{G}_\mathcal{Y}$ is invertible in the quotient algebra $\mathcal{F}_\mathcal{Y}/\mathcal{G}_\mathcal{Y}$. Thus, every stability problem is equivalent to an invertibility problem in a suitably chosen C^*-algebra.

Let further stand $\mathcal{F}_\mathcal{Y}^C$ for the set of all sequences $\mathbf{A} = (A_n)$ of operators $A_n : \operatorname{im} P_{Y_n} \to \operatorname{im} P_{Y_n}$ with the property that the sequences $(A_n P_{Y_n})$ and $(A_n^* P_{Y_n})$ converge strongly. By the uniform boundedness principle, the quantity $\sup \|A_n P_{Y_n}\|$ is finite for every sequence (A_n) in $\mathcal{F}_\mathcal{Y}^C$. Thus, $\mathcal{F}_\mathcal{Y}^C$ is a closed and symmetric subalgebra of $\mathcal{F}_\mathcal{Y}$ which contains $\mathcal{G}_\mathcal{Y}$, and the mapping

$$W : \mathcal{F}_\mathcal{Y}^C \to L(l^2(X)), \quad (A_n) \mapsto \text{s-lim}\, A_n P_{Y_n} \tag{3.1}$$

is a *-homomorphism. Note that $\mathbf{I} \in \mathcal{F}_\mathcal{Y}^C$ and that $W(\mathbf{I})$ is the identity operator I on $L^2(\Gamma)$.

For each C^*-subalgebra A of $L(l^2(\Gamma))$, write D for the mapping of finite sections (or spatial) discretization, i.e.,

$$D : L(l^2(\Gamma)) \to \mathcal{F}_\mathcal{Y}, \quad A \mapsto (P_{Y_n} A P_{Y_n}), \tag{3.2}$$

and let $\mathcal{S}_\mathcal{Y}(\mathsf{A})$ stand for the smallest closed C^*-subalgebra of the algebra $\mathcal{F}_\mathcal{Y}$ which contains all sequences $D(A)$ with $A \in \mathsf{A}$. Clearly, $\mathcal{S}_\mathcal{Y}(\mathsf{A})$ is contained in $\mathcal{F}_\mathcal{Y}^C$, and the mapping W in (3.1) induces a *-homomorphism from $\mathcal{S}_\mathcal{Y}(\mathsf{A})$ onto A. On this level, one cannot say much about the algebra $\mathcal{S}_\mathcal{Y}(\mathsf{A})$. The simple proof of the following is in [14].

Proposition 3.1. *Let A be a C^*-subalgebra of $L(l^2(\Gamma))$. Then the finite sections discretization $D : \mathsf{A} \to \mathcal{F}_\mathcal{Y}$ is an isometry, and $D(\mathsf{A})$ is a closed subspace of the algebra $\mathcal{S}_\mathcal{Y}(\mathsf{A})$. This algebra splits into the direct sum*

$$\mathcal{S}_\mathcal{Y}(\mathsf{A}) = D(\mathsf{A}) \oplus (\ker W \cap \mathcal{S}_\mathcal{Y}(\mathsf{A})),$$

and for every operator $A \in \mathsf{A}$ one has

$$\|D(A)\| = \min_{K \in \ker W} \|D(A) + K\|.$$

Finally, $\ker W \cap \mathcal{S}_\mathcal{Y}(\mathsf{A})$ is equal to the quasicommutator ideal of $\mathcal{S}_\mathcal{Y}(\mathsf{A})$, i.e., to the smallest closed ideal of $\mathcal{S}_\mathcal{Y}(\mathsf{A})$ which contains all sequences $(P_{Y_n} A_1 P_{Y_n} A_2 P_{Y_n} - P_{Y_n} A_1 A_2 P_{Y_n})$ with operators A_1, $A_2 \in \mathsf{A}$.

We denote the ideal $\ker W \cap \mathcal{S}_{\mathcal{Y}}(\mathsf{A})$ by $\mathcal{J}_{\mathcal{Y}}(\mathsf{A})$. Since the first item in the decomposition $D(\mathsf{A}) \oplus \mathcal{J}_{\mathcal{Y}}(\mathsf{A})$ of $\mathcal{S}_{\mathcal{Y}}(\mathsf{A})$ is isomorphic (as a linear space) to A, a main part of the description of the algebra $\mathcal{S}_{\mathcal{Y}}(\mathsf{A})$ is to identify the ideal $\mathcal{J}_{\mathcal{Y}}(\mathsf{A})$.

We are going to present two alternate descriptions of the quasicommutator ideal $\mathcal{J}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$ of the finite sections algebra $\mathcal{S}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$. For we have to introduce some notions of topological type. Note that the standard topology on Γ is the discrete one; so every subset of Γ is open with respect to this topology.

Let Ω be a finite subset of Γ which contains the identity element e and which generates Γ as a semi-group, i.e., if we set $\Omega_0 := \{e\}$ and if we let Ω_n denote the set of all words of length at most n with letters in Ω for $n \geq 1$, then $\cup_{n \geq 0} \Omega_n = \Gamma$. Note also that the sequence (Ω_n) is increasing; so the operators P_{Ω_n} can play the role of the finite sections projections P_{Y_n}, and in fact we will obtain some of the subsequent results exactly for this sequence.

With respect to Ω, we define the following "algebro-topological" notions. Let $A \subseteq \Gamma$. A point $a \in A$ is called an Ω-*inner* point of A if $\Omega a := \{\omega a : \omega \in \Omega\} \subseteq A$. The set $\mathrm{int}_\Omega A$ of all Ω-inner points of A is called the Ω-*interior* of A, and the set $\partial_\Omega A := A \setminus \mathrm{int}_\Omega A$ is the Ω-*boundary* of A. Note that we consider the Ω-boundary of a set always as a part of that set. (In this point, the present definition of a boundary differs from other definitions in the literature; see [1] for instance.)

One easily checks that the Ω-interior and the Ω-boundary of a set are invariant with respect to multiplication from the right-hand side:

$$(\mathrm{int}_\Omega A)s = \mathrm{int}_\Omega (As) \quad \text{and} \quad (\partial_\Omega A)s = \partial_\Omega (As)$$

for $s \in \Gamma$. One also has

$$\Omega_{n-1} \subseteq \mathrm{int}_\Omega \Omega_n \subseteq \Omega_n \quad \text{for each } n \geq 1, \tag{3.3}$$

whence

$$\partial_\Omega \Omega_n \subseteq \Omega_n \setminus \Omega_{n-1} \quad \text{for each } n \geq 1. \tag{3.4}$$

Here is a first result which describes $\mathcal{J}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$ in terms of generators of Γ. Abbreviate $I - P_A =: Q_A$.

Theorem 3.2. *$\mathcal{J}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$ is the smallest closed ideal of $\mathcal{S}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$ which contains all sequences*

$$(P_{Y_n} L_{\omega^{-1}} Q_{Y_n} L_\omega P_{Y_n})_{n \geq 1} \quad \textit{with} \quad \omega \in \Omega. \tag{3.5}$$

We call $(P_{\partial_\Omega Y_n})_{n \geq 1}$ the *sequence of the discrete boundaries* of the finite section method with respect to (Y_n). Note that the assumptions in the following theorem are satisfied if $Y_n = \Omega_n$ due to (3.3).

Theorem 3.3. *Assume that $Y_{n-1} \subseteq \mathrm{int}_\Omega Y_n \subseteq Y_n$ for all $n \geq 2$ and that $\cup_{n \geq 1} Y_n = \Gamma$. Then the sequence $(P_{\partial_\Omega Y_n})_{n \geq 1}$ of the discrete boundaries belongs to the algebra $\mathcal{S}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$, and the quasicommutator ideal is generated by this sequence, i.e., $\mathcal{J}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$ is the smallest closed ideal of $\mathcal{S}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$ which contains $(P_{\partial_\Omega Y_n})_{n \geq 1}$.*

Both results were proved in [15] for the ideal $\mathcal{J}_{\mathcal{Y}}(\mathsf{Sh}(\Gamma))$ of $\mathcal{S}_{\mathcal{Y}}(\mathsf{Sh}(\Gamma))$ in place of $\mathcal{J}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$. The above theorems follow from these results since every multiplication operator aI commutes with every projection P_Y where $Y \subseteq \Gamma$.

4. Stability

We are now going to study the stability of sequences in $\mathcal{S}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$ via the limit operators method. The key observations are that the stability of a sequence in that algebra is equivalent to the Fredholm property of a certain associated operator, which is band-dominated, such that the Fredholm property of that operator can be studied by means of its limit operators via Roe's result.

Let again $\mathcal{Y} := (Y_n)$ be an increasing sequence of finite subsets of Γ with $\cup_{n \geq 1} Y_n = \Omega$. A sequence $(v_n) \subseteq \Gamma$ is called an *inflating* sequence for $\mathcal{Y}$ if $Y_m v_m^{-1} \cap Y_n v_n^{-1} = \emptyset$ for $m \neq n$. The existence of inflating sequences is easy to see. Moreover, the following lemma was shown in [15].

Lemma 4.1. *Let $\mathcal{Y} = (Y_n)$ be as above and V an infinite subset of Γ. Then there is an inflating sequence for $\mathcal{Y}$ in V.*

In what follows we choose and fix an inflating sequence (v_n) for $\mathcal{Y}$ and set

$$\Gamma' := \Gamma \setminus \cup_{n=1}^{\infty} Y_n v_n^{-1}. \tag{4.1}$$

For $s \in \Gamma$, let again $R_s : l^2(\Gamma) \to l^2(\Gamma)$ refer to the operator $(R_s f)(t) := f(ts)$. Evidently, $R_s L_t = L_t R_s$ for $s, t \in \Gamma$. The proof of the following theorem is in [15].

Theorem 4.2. *Let $\mathbf{A} = (A_n) \in \mathcal{F}_{\mathcal{Y}}$. Then*

(a) *the series*

$$\sum_{n=1}^{\infty} R_{v_n} A_n R_{v_n}^{-1} \tag{4.2}$$

converges strongly on $l^2(\Gamma)$. The sum of this series is denoted by $\operatorname{Op}(\mathbf{A})$.

(b) *the sequence (A_n) is stable if and only if the operator $\operatorname{Op}(\mathbf{A}) + P_{\Gamma'}$ is Fredholm on $l^2(\Gamma)$.*

(c) *The mapping Op is a continuous homomorphism from $\mathcal{F}_{\mathcal{Y}}$ to $L(l^2(\Gamma))$.*

The applicability of Roe's result to the study the stability of the finite section method for band-dominated operators rests of the following fact.

Proposition 4.3. *Let $\mathbf{A}$ be a sequence in $\mathcal{S}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$. Then $\operatorname{Op}(\mathbf{A})$ is a band-dominated operator.*

Proof. First let $A \in \mathsf{BDO}(\Gamma)$ be a band operator and let Γ_0 be a band width of A. It is easy to check that then $R_{v_n} P_{Y_n} A P_{Y_n} R_{v_n}^{-1}$ is a band operator with the same band width for every n. The inflating property ensures that $\operatorname{Op}((P_{Y_n} A P_{Y_n}))$ is a band operator with band width Γ_0, too. Now Theorem 4.2 (c) yields the assertion. □

In order to verify the stability of a sequence $\mathbf{A} \in \mathcal{S}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$ via the above results, we thus have to compute the limit operators of $\mathrm{Op}\,(\mathbf{A}) + P_{\Gamma'}$, which will be our next goal. Note that the exactness of Γ is not relevant in this computation.

Let Ω be a finite subset of Γ with $e \in \Omega$ which generates Γ as a semi-group and define Ω_n as above. By Theorem 4.2, the Fredholm property of an operator $\mathrm{Op}\,(\mathbf{A})$ is independent of the concrete choice of the inflating sequence. For technical reasons, we choose an inflating sequence (v_n) for the sequence

$$\big((Y_n \cup \Omega_n)(Y_n \cup \Omega_n)^{-1}(Y_n \cup \Omega_n)\big)_{n\geq 1}$$

instead of $(Y_n)_{n\geq 1}$. Since

$$Y_n \cup \Omega_n \subset (Y_n \cup \Omega_n)(Y_n \cup \Omega_n)^{-1} \subset (Y_n \cup \Omega_n)(Y_n \cup \Omega_n)^{-1}(Y_n \cup \Omega_n),$$

(v_n) is also an inflating sequence for (Y_n). Moreover, since $\text{s-lim}\, P_{\Omega_n} = P_\Gamma = I$, one also has

$$\text{s-lim}\, P_{(Y_n \cup \Omega_n)(Y_n \cup \Omega_n)^{-1}} = P_\Gamma = I. \tag{4.3}$$

Let now $\mathbf{A} = (A_n) \in \mathcal{S}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$, set as before

$$\mathrm{Op}\,(\mathbf{A}) = \sum_{n=1}^{\infty} R_{v_n} A_n R_{v_n}^{-1} \quad \text{and} \quad \Gamma' = \Gamma \setminus \cup_{n=1}^{\infty} Y_n v_n^{-1},$$

and let $h : \mathbb{N} \to \Gamma$ be a sequence tending infinity for which the limit operator

$$(\mathrm{Op}\,(\mathbf{A}) + P_{\Gamma'})_h := \text{s-lim}_{n\to\infty} R_{h(n)}^{-1} (\mathrm{Op}\,(\mathbf{A}) + P_{\Gamma'}) R_{h(n)}$$

exists. Then the limit operator $(\mathrm{Op}\,(\mathbf{A}) + P_{\Gamma'})_g$ exists for every subsequence g of h, and it coincides with $(\mathrm{Op}\,(\mathbf{A}) + P_{\Gamma'})_h$. So we can freely pass to subsequences of h if necessary. By a first passage to a suitable subsequence of h we can arrange that one of the following two situations happens; so we can restrict the computation of the limit operator to these cases:

Case 1: All elements $h(n)$ belong to $\cup_{k\geq 1} v_k Y_k^{-1}$.

Case 2: No element $h(n)$ belongs to $\cup_{k\geq 1} v_k Y_k^{-1}$.

We start with **Case 1**. Passing again to a subsequence of h, if necessary, we can further suppose that each $h(n)$ belongs to one of the sets $v_k Y_k^{-1}$, say to $v_{k_n} Y_{k_n}^{-1}$, and that $v_{k_n} Y_{k_n}^{-1}$ contains no other element of the sequence h besides $h(n)$. For each n, let r_n denote the smallest non-negative integer such that $h(n) \in v_{k_n} (\partial_\Omega Y_{k_n})^{-1} \Omega_{r_n}$. Thus, r_n measures the distance of $h(n)$ to the Ω-boundary of $v_{k_n} Y_{k_n}^{-1}$. Set $r^* := \liminf_{n\to\infty} r_n$. Again we have to distinguish two cases, namely when r^* is finite and when r^* is infinite. We refer to these cases as Case 1.1 and 1.2, respectively. Then Theorems 4.4 and 4.6 below can be derived in the similar way as the corresponding Theorems 4.9 and 4.11 in [15], with some evident modifications.

Theorem 4.4. *Let* $\mathbf{A} \in \mathcal{S}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$*, and let* h *be a sequence such that the limit operator of* $\mathrm{Op}\,(\mathbf{A}) + P_{\Gamma'}$ *with respect to* h *exists. In* Case 1.1*, there is a subsequence* g *of* h *such that the limit operator* $(P_{\Gamma'})_g$ *exists, and there are a monotonically* *increasing sequence* (k_n) *in* $\mathbb{N}$*, a vector* $\eta_{k_n} \in (\partial_\Omega Y_{k_n})^{-1}$ *for each* $n \geq 1$*, and a* $w_* \in \Gamma$ *such that*

$$(\mathrm{Op}\,(\mathbf{A}) + P_{\Gamma'})_h = \text{s-lim}\, R_{w_*}^{-1} R_{\eta_{k_n}}^{-1} A_{k_n} R_{\eta_{k_n}} R_{w_*} + (P_{\Gamma'})_g.$$

Thus, the operator A_{k_n} living on $\mathrm{im}\, P_{Y_{k_n}}$ is shifted by a vector $\eta_{k_n} \in (\partial_\Omega Y_{k_n})^{-1}$ and by another vector w_* independent of n. It is only a matter of taste to consider A_{k_n} as shifted by the vector $\eta_{k_n}^{-1}$ belonging to the Ω-boundary of Y_{k_n}. In particular, every limit operator of $\mathrm{Op}\,(\mathbf{A})$ is a shift by some vector w_* of a strong limit of operators A_{k_n}, shifted by vectors in the Ω-boundary of Y_{k_n}. This fact is well known for the group $\mathbb{Z}$ and intervals $Y_k = [-k, k] \cap \mathbb{Z}$ (and has been employed in [12] to get rid of the uniform boundedness condition in this case), and it was observed by Lindner [6] in case $\Gamma = \mathbb{Z}^N$ and $Y_k = \Omega_k$ is a convex polygon with integer vertices.

Before turning to the other cases, let us specify Theorem 4.4 to pure finite sections sequences for operators in $\mathsf{BDO}(\Gamma)$. The existence of the limit operator $(P_{\Gamma'})_h$ is guaranteed if the strong limit

$$\text{s-lim}\, R_{w_*}^{-1} R_{\eta_{k_n}}^{-1} P_{Y_{k_n}} R_{\eta_{k_n}} R_{w_*} = \text{s-lim}\, P_{Y_{k_n} \eta_{k_n} w_*} \tag{4.4}$$

exists. In this case, there is a subset $\mathcal{Y}^{(h)}$ of Γ such that

$$\text{s-lim}\, P_{Y_{k_n} \eta_{k_n} w_*} = P_{\mathcal{Y}^{(h)}} \tag{4.5}$$

and, thus, $(P_{\Gamma'})_g = I - P_{\mathcal{Y}^{(h)}}$. We claim that the sequence $(\eta_{k_n} w_*)_{n \geq 1}$ tends to infinity. For this goal, it is sufficient to show that every sequence (μ_n) with $\mu_n \in \partial_\Omega Y_{k_n}$ tends to infinity. Let Γ_0 be a finite subset of Γ. Choose n_0 such that $\Gamma_0 \subseteq \Omega_{n_0 - 1}$ and n^* such that $\Omega_{n_0} \subseteq Y_{k_n}$ for all $n \geq n^*$. Then $\mathrm{int}_\Omega \Omega_{n_0} \subseteq \mathrm{int}_\Omega Y_{k_n}$, and from (3.3) we conclude that

$$\Gamma_0 \subseteq \Omega_{n_0 - 1} \subseteq \mathrm{int}_\Omega \Omega_{n_0} \subseteq \mathrm{int}_\Omega Y_{k_n}.$$

Hence, $\partial_\Omega Y_{k_n} \cap \Gamma_0 = \emptyset$ for all $n \geq n^*$, whence the claimed convergence.

Given a sequence h such that the limit (4.4) exists and a band-dominated operator A, let $\sigma_{op,\,h}(A)$ denote the set of all limit operators of A with respect to subsequences of the sequence $(\eta_{k_n} w_*)_{n \geq 1}$. This set is not empty by Proposition 2.3.

Proposition 4.5. *Let* $A \in \mathsf{BDO}(\Gamma)$*, and let* h *be a sequence such that the limit operator* $\mathrm{Op}\,(\mathbf{A})_h$ *for the sequence* $(P_{Y_n} A P_{Y_n})$ *exists. In* Case 1.1*, there are* k_n*,* η_{k_n} *and* w_* *as in Theorem* 4.4 *such that the limit* (4.4) *exists. Then there is a limit operator* $A_g \in \sigma_{op,\,h}(A)$ *of* A *such that*

$$(\mathrm{Op}\,(\mathbf{A}) + P_{\Gamma'})_h = P_{\mathcal{Y}^{(h)}} A_g P_{\mathcal{Y}^{(h)}} + (I - P_{\mathcal{Y}^{(h)}}). \tag{4.6}$$

Conversely, if the limit (4.4) *exists for a certain choice of* k_n*,* η_{k_n} *and* w_* *as in Theorem* 4.4 *and if* A_g *is a limit operator of* A *with respect to a certain subsequence* $g = (\eta_{k_{n_r}} w_*)_{r \geq 1}$ *of the sequence* $(\eta_{k_n} w_*)_{n \geq 1}$*, then the limit operator* $\mathrm{Op}\,(\mathbf{A})_h$ *exists for the sequence* $h = (v_{k_{n_r}} g_r)_{r \geq 1}$*, and* (4.6) *holds.*

Proof. The proof of the first assertion follows easily from Theorem 4.4. Indeed,

$$R_{w_*^{-1}\eta_{k_n}^{-1}} P_{Y_{k_n}} A P_{Y_{k_n}} R_{\eta_{k_n} w_*} = (R_{w_*^{-1}\eta_{k_n}^{-1}} P_{Y_{k_n}} R_{\eta_{k_n} w_*}) \cdot (R_{w_*^{-1}\eta_{k_n}^{-1}} A R_{\eta_{k_n} w_*}) \cdot (R_{w_*^{-1}\eta_{k_n}^{-1}} P_{Y_{k_n}} R_{\eta_{k_n} w_*}).$$

The sequences in the outer parentheses converge strongly to $P_{\mathcal{Y}^{(h)}}$. If now g is a subsequence of $(\eta_{k_n} w_*)_{n\geq 1}$ such that the limit operator A_g exists, then we conclude that

$$R_{w_*^{-1}\eta_{k_n}^{-1}} P_{Y_{k_n}} A P_{Y_{k_n}} R_{\eta_{k_n} w_*} \to P_{\mathcal{Y}^{(h)}} A_g P_{\mathcal{Y}^{(h)}}$$

*-strongly as $n \to \infty$. The second assertion is evident. □

Now we turn over to Case 1.2, when r^* is infinite.

Theorem 4.6. *Let* $\mathbf{A} \in \mathcal{S}_{\mathcal{Y}}(\mathsf{BDO}(\Gamma))$ *and* $A := \text{s-lim} A_n P_{Y_n}$, *and let* h *be a sequence such that the limit operator* $\mathrm{Op}\,(\mathbf{A})_h$ *exists. Then, in* Case 1.2, *either* $\mathrm{Op}\,(\mathbf{A})_h = R_{v^*}^{-1} A R_{v^*}$ *with a fixed* $v^* \in \Gamma$, *or there is a limit operator* A_g *of* A *such that* $\mathrm{Op}\,(\mathbf{A})_h = A_g$. *Conversely, each operator* $R_{v^*}^{-1} A R_{v^*}$ *with* $v^* \in \Gamma$ *and each limit operator* A_g *of* A *occur as limit operators of* $\mathrm{Op}\,(\mathbf{A})$.

Proof. It is sufficient to verify the assertion for pure finite sections sequences $\mathbf{A} = (P_{Y_n} A P_{Y_n})$ with $A \in \mathsf{BDO}(\Gamma)$. For these sequences, one has

$$\begin{aligned} R_{h(n)}^{-1}(\mathrm{Op}\,(\mathbf{A}) + P_{\Gamma'}) R_{h(n)} &= \sum_{k\neq k_n} R_{h(n)}^{-1} R_{v_k} P_{Y_k} A P_{Y_k} R_{v_k}^{-1} R_{h(n)} (I - P_{Y_{k_n} v_{k_n}^{-1} h(n)}) \\ &\quad + R_{h(n)}^{-1} P_{\Gamma'} R_{h(n)} (I - P_{Y_{k_n} v_{k_n}^{-1} h(n)}) \\ &\quad + P_{Y_{k_n} v_{k_n}^{-1} h(n)} (R_{h(n)}^{-1} R_{v_{k_n}} A R_{v_{k_n}}^{-1} R_{h(n)}) P_{Y_{k_n} v_{k_n}^{-1} h(n)}. \end{aligned}$$

Consider the sequence $(v_{k_n}^{-1} h(n))$, which is either finite or contains a subsequence which tends to infinity. In the first case, there is a $v^* \in \Gamma$ which is met by this sequence infinitely often, whence $\mathrm{Op}\,(\mathbf{A})_h = R_{v^*} A R_{v^*}^{-1}$. In the second case, Proposition 2.3 implies the existence of a subsequence g of $(v_{k_n}^{-1} h(n))$ which tends to infinity and for which the limit operator A_g exists. In this case, $\mathrm{Op}\,(\mathbf{A})_h = A_g$.

Conversely, given $v^* \in \Gamma$ and a limit operator A_g of A, one can choose $h(n) := v_{k_n} v^*$ and $h(n) := v_{k_n} g(n)$ in order to obtain the limit operators $R_{v^*}^{-1} A R_{v^*}$ and A_g of $\mathrm{Op}\,(\mathbf{A})$, respectively. □

Note that, in Case 1.2, the invertibility of all limit operators of $\mathrm{Op}\,(\mathbf{A})$ as well as the uniform boundedness of the norms of their inverses follows already from the invertibility of A.

Now consider **Case 2**, i.e., suppose that none of the $h(n)$ belongs to $\cup v_k Y_k^{-1}$. For $n \in \mathbb{N}$, let r_n stand for the smallest non-negative integer such that there is a $k_n \in \mathbb{N}$ with $h(n) \in v_{k_n} (\partial_\Omega Y_{k_n})^{-1} \Omega_{r_n}$. Consequently,

$$h(n) \notin v_{k_n} (\partial_\Omega Y_{k_n})^{-1} \Omega_{r_n - 1} \quad \text{for all } n.$$

Again we set $r^* := \liminf r_n$ and distinguish the cases, when r^* is finite and when r^* is infinite, to which we refer as Case 2.1 and 2.2, respectively. These cases can be studied in a similar way as the corresponding cases in [15], and we again omit the details. The following theorem summarizes the results from Cases 1.1–2.2.

Theorem 4.7. *Let $\mathbf{A} \in \mathcal{S}_\mathcal{Y}(\mathsf{BDO}(\Gamma))$ and $A :=$ s-lim $A_n P_{Y_n}$. Then the limit operators of* $\mathrm{Op}\,(\mathbf{A}) + P_{\Gamma'}$ *are the identity operator I, all shifts $R_{v^*}^{-1} A R_{v^*}$ of the operator A, all limit operators of A, and all operators of the form*

$$\text{s-lim}\, R_{w_*}^{-1} R_{\eta_{k_n}}^{-1} A_{k_n} R_{\eta_{k_n}} R_{w_*} + (P_{\Gamma'})_g$$

with a suitable subsequence g of h and with elements $\eta_{k_n} \in (\partial_\Omega Y_{k_n})^{-1}$ and $w_* \in \Gamma$.

Combining this theorem with Theorems 4.2 (b), 2.4 and 2.5 we arrive at the following stability results.

Theorem 4.8. *Let Γ be a finitely generated exact discrete group, and let $(A_n) \in \mathcal{S}_\mathcal{Y}(\mathsf{BDO}(\Gamma))$. The sequence (A_n) is stable if and only if the operator*

$$A := \text{s-lim}\, A_n P_{Y_n}$$

and all operators of the form

$$\text{s-lim}\, R_{\eta_{k_n}}^{-1} A_{k_n} R_{\eta_{k_n}} + R_{w_*} (P_{\Gamma'})_g R_{w_*}^{-1}$$

with a suitable subsequence g of h and with elements $\eta_{k_n} \in (\partial_\Omega Y_{k_n})^{-1}$ and $w_* \in \Gamma$ are invertible and if the norms of their inverses are uniformly bounded.

Theorem 4.9. *Let Γ be a finitely generated exact discrete group, and let $A \in \mathsf{BDO}(\Gamma)$. The sequence $\mathbf{A} = (P_{Y_n} A P_{Y_n})$ is stable if and only if the operator A and all operators*

$$P_{\mathcal{Y}^{(h)}} A_g P_{\mathcal{Y}^{(h)}} : \mathrm{im}\, P_{\mathcal{Y}^{(h)}} \to \mathrm{im}\, P_{\mathcal{Y}^{(h)}}$$

where h is a sequence such that the limit (4.4) *exists and $\mathcal{Y}^{(h)}$ is as in* (4.5) *and where g is in $\sigma_{op,\,h}(A)$ are invertible and if the norms of their inverses are uniformly bounded.*

Theorem 4.10. *Let Γ be a finitely generated discrete and exact group with subexponential growth which possesses at least one non-cyclic element, and let A be a band operator on $l^2(\Gamma)$. Then the sequence $\mathbf{A} = (P_{Y_n} A P_{Y_n})$ is stable if and only if the operators mentioned in the previous theorem are invertible.*

There are special sequences $\mathcal{Y} = (Y_n)$ and $\eta : \mathbb{N} \to \Gamma$ for which the existence of the limit (4.5) can be guaranteed. Let again Ω_n refer to the set of all products of at most n elements of Ω and set $\Omega_0 := \{e\}$. A sequence (ν_n) in Γ is called a *geodesic ray* (with respect to Ω) if there is a sequence (w_n) in $\Omega \setminus \{e\}$ such that $\nu_n = w_1 w_2 \ldots w_n$ and $\nu_n \in \Omega_n \setminus \Omega_{n-1}$ for each $n \geq 1$. Note that this condition implies that each ν_n is in the *right Ω-boundary of* Ω_n, which is the set of all $w \in \Omega_n$ for which $w\Omega$ is not a subset of Ω_n.

We will see now that the $\lim \Omega_n \eta_n$ exists if η is an inverse geodesic ray, i.e., if $\eta_n = \nu_n^{-1}$ for a geodesic ray ν.

Lemma 4.11. *Let $(w_n)_{n\geq 1}$ be a sequence in Ω and set $\eta_n := w_n^{-1} w_{n-1}^{-1} \dots w_1^{-1}$ for $n \geq 1$. Then the strong limit* s-lim $P_{\Omega_n \eta_n}$ *exists, and*

$$\text{s-lim}\, P_{\Omega_n \eta_n} = P_{\cup_{n\geq 1} \Omega_n \eta_n}. \tag{4.7}$$

Proof. For $n \geq 1$, one has $\Omega_n \eta_n = \Omega_n w_{n+1} w_{n+1}^{-1} w_n^{-1} \dots w_1^{-1} \subseteq \Omega_{n+1} \eta_{n+1}$. These inclusions imply the existence of the strong limit and the equality (4.7). □

The natural question arises whether every sequence $\eta : \mathbb{N} \to \Gamma$ for which the limit (4.5) exists has a subsequence which is a subsequence of an inverse geodesic ray. If the answer is affirmative, then it would prove sufficient to consider strong limits with respect to inverse geodesic rays in Theorem 4.8. Under some conditions, this question was answered in [15] for commutative groups Γ and for the free (non-commutative) groups $\mathbb{F}_N$ with N generators.

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Joint Defect Index of a Cyclic Tuple of Symmetric Operators

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Abstract. Von Neumann showed that the defect indices of a symmetric operator are invariant in each the upper half and lower half complex planes, and if the operator commutes with a conjugation operator, the indices have the same value in $\mathbb{C}\backslash\mathbb{R}$. This leads to self-adjoint extensions for the operator. We prove an anlogous invariance result in $\mathbb{C}^d\backslash\mathbb{R}^d$ for a class of operator tuples. We also apply this to give a result regarding reproducing kernels.

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1. Introduction

John von Neumann's theory of defect indices gives conditions necessary and sufficient to determine if a single unbounded symmetric operator on Hilbert space has self-adjoint extensions and a method to parametrize them all. In the case of a tuple of symmetric operators much less is known. Given a tuple of symmetric operators $(A_1, \dots, A_d)$ on a Hilbert space $\mathcal{H}$, there exists a joint projection-valued spectral measure if and only if there exists a Hilbert space extension $\mathcal{K} \supseteq \mathcal{H}$ and self adjoint extensions $B_j \supset A_j$ on $\mathcal{K}$ whose spectral projections commute.

The issue of commutativity is a large obstacle to the generalization of spectral theory of a single operator to a the spectral theory of a tuple of operators. Nelson's example [7] shows that two operators may not strongly commute, even if they do commute on a common core. In the context of the multi-dimensional moment problem, [3] and [10] concurrently gave examples of tuples of operators which weakly commute on a domain dense in a Hilbert space, yet the tuple possesses no strongly commuting self-adjoint extension. Implicit in both approaches is that such a commuting self-adjoint extension exists if and only if the corresponding moment

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problem has a solution [5]. Since not every positive polynomial in two or more variables may be written as a sum of squares, a Riesz extension theorem argument and the Riesz-Haviland theorem gives a construction for a tuple of operators which weakly commute on a dense subspace of a Hilbert space but do not have a joint self-adjoint extension as above.

2. Formulation and Notation

We have a choice of starting points. In the case of the multi-dimensional moment problem an operator tuple arises naturally in the GNS construction. In the case of a tuple of symmetric operators, we make assumptions to align these two cases.

2.1. The multi-dimensional moment problem

Let $\{s_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ be a multisequence indexed by the monomials in d variables. We use the standard multivariable notation and say $x = (x_1,\dots,x_d) \in \mathbb{R}^d$, $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}_0^d$, $x^\alpha = x_1^{\alpha_1}\cdots x_d^{\alpha_d}$, and $|\alpha| = \alpha_1+\cdots+\alpha_d$.

Define the Riesz functional $L : \mathbb{C}[x] \to \mathbb{C}$ for the multisequence by $L(x^\alpha) = s_\alpha$, and extend linearly. The sequence $\{s_\alpha\}$ and the functional L are called *positive* if $L(|p|^2) \geq 0$ for every $p \in \mathbb{C}[x]$. In this case, we can define a positive semidefinite sesquilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{C}[x]$ by $\langle p,q\rangle = L(p\overline{q})$. We then use the GNS construction to pass to an inner product $\langle\cdot,\cdot\rangle$ on a Hilbert space $\mathcal{H}$ such that the polynomials are dense in $\mathcal{H}$. This construction is carried out in more detail in [5]. We will generally make no distinction between a polynomial $p(x)$ and its representative in $\mathcal{H}$.

For $1 \leq j \leq d$, let X_j be multiplication by the real variable x_j on $\mathbb{C}[x]$. Since X_j is symmetric, let $\overline{X}_j$ be the closure of X_j and $\mathcal{D}_j \subset \mathcal{H}$ be the domain of $\overline{X}_j$.

2.2. A cyclic tuple

Equivalently suppose that $(X_1,\dots,X_d)$ is a tuple of symmetric operators on a Hilbert space $\mathcal{H}$ and that there exists a conjugation operator $C : \mathcal{H} \to \mathcal{H}$ which leaves the domain of each X_j invariant and commutes with it. Furthermore, suppose that there is a cyclic vector $\xi \in \mathcal{H}$ for the tuple, and that for any $1 \leq i,j \leq d$, X_i and X_j commute on $\{p(X)\xi : p(x) \in \mathbb{C}[x]\}$. We use the standard notation that $p(X)$ is the evaluation of $p(x)$ at $(X_1,\dots,X_d)$.

If we consider the minimal domain $\{p(X)\xi : p \in \mathbb{C}[x]\}$ for each X_j and $\mathcal{D}_j \subset \mathcal{H}$ be the domain of $\overline{X}_j$ as previously, then we are nearly in the situation above. The remaining difference is that the vector 1 is invariant under complex conjugation in the moment problem setting, so we also assume that $C\xi = \xi$.

The assumption that there is a conjugation operator which commutes with the operator tuple is not unreasonable for a formally commuting tuple. In the case where the tuple consists of strongly commuting self-adjoint operators, let E be the joint projection-valued measure on $\mathbb{R}^d$. Then our conjugation C is complex conjugation on $L^2(E)$.

2.3. Continuation

Consider the operator $X : \mathbb{C}[x]^d \to \mathcal{H}$, defined for $p = (p_1, \dots, p_d) \in \mathbb{C}[x]^d$ by

$$Xp(x) = x_1 p_1(x) + x_2 p_2(x) + \cdots + x_d p_d(x).$$

Denote the operator $\hat{X} : \bigoplus_{j=1}^d \mathcal{D}_j \to \mathcal{H}$ by

$$\hat{X} f = \overline{X}_1 f_1 + \cdots + \overline{X}_d f_d,$$

where $f = (f_1, \dots, f_d) \in \bigoplus_{j=1}^d \mathcal{D}_j$.

On the set $\mathbb{C}[x]$ the adjoint is formally defined as $X^* p = (x_1 p, \dots, x_d p)$. This element induces a bounded linear functional on $\mathcal{H}^d$ since

$$|\langle (f_1, \dots, f_d), (x_1 p, \dots, x_d p) \rangle| \leq \|(f_1, \dots, f_d)\| \|(x_1 p, \dots, x_d p)\|$$

for all $f = (f_1, \dots, f_d) \in \mathcal{H}^d$. Since $\mathbb{C}[x]$ is dense in $\mathcal{H}$, it follows that the adjoint X^* is well defined and has dense domain. Thus the closure $\overline{X}$ of X exists and $X^{**} = \overline{X}$.

By considering the isometry

$$(p(x), x_k p(x)) \mapsto (0, \dots, 0, \underset{k^{\text{th}}\ \text{place}}{\underset{\uparrow}{p(x)}}, 0, \dots, 0, x_k p(x))$$

from the graph of X_j into the graph of X, this induces an isometry from the graph of $\overline{X}_j$ into the graph of $\overline{X}$. Therefore we conclude that $\hat{X} \subset \overline{X}$.

3. Main result

For $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, define

$$(X - z)(p_1, \dots, p_d) = (x_1 - z_1) p_1 + \cdots + (x_d - z_d) p_d,$$

and define $\hat{X} - z$ and $\overline{X} - z$ analogously. Since $\overline{X}$ is a closed operator, for any $z \in \mathbb{C}^d$, $\ker(\overline{X} - z)$ is a closed subspace of $\mathcal{H}^d$. Thus we can decompose its domain orthogonally into two subspaces. Define $\mathcal{E}_z = \mathcal{D}_{\overline{X}} \ominus \ker(\overline{X} - z)$, and call $\mathcal{E}_z$ an *effective domain* of $\overline{X} - z$. In particular, $\overline{X} - z|_{\mathcal{E}_z}$ is bijective onto the range of $\overline{X} - z$. We will show that if $z \in \mathbb{C}^d \backslash \mathbb{R}^d$, and if $w - z$ is sufficiently small, then $\overline{X} - w$ is bijective from $\mathcal{E}_z$ to the range of $\overline{X} - w$.

Let $\mathcal{E}'_z = \mathcal{E}_z \cap \mathbb{C}[x]^d = \{(x_1 - \overline{z_1}) p, (x_2 - \overline{z_2}) p, \dots, (x_d - \overline{z_d}) p | p \in \mathbb{C}[x]\}$ be an effective domain of X. The second equality follows from that $\operatorname{ran}((X - z)^*)$ is a dense subspace of $\mathcal{H}^d \ominus \ker(\overline{X} - z)$ and that $\mathcal{E}'_z \subseteq \mathbb{C}[x]^d$. Since $\overline{X}$ is the closure of X, we will examine $\mathcal{E}'_z$ to explore $\overline{X}$.

Lemma 1. *If $z \in \mathbb{C}^d \backslash \mathbb{R}^d$, then $\overline{X} - z$ has closed range.*

Proof. Let $z = (z_1 \dots, z_d) \in \mathbb{C}^d \backslash \mathbb{R}^d$, where $z_j = a_j + i b_j$. To show that $\overline{X} - z$ has closed range, we will show that $X - z$ is bounded from below on $\mathcal{E}'_z$. Let Let

$f = ((x_1 - \overline{z_1})p, \dots, (x_d - \overline{z_d})p) \in \mathcal{E}_z'$. Then

$$\|f\|^2 = \left\| \left(\sum_{j=1}^{d} (x_j - a_j)^2 + \sum_{k=1}^{d} b_k^2 \right) |p|^2 \right\|^2 .$$

Denoting $|x|^2 = \sum_{j=1}^{d} x_j^2$, we compute:

$$\begin{aligned} \|(X - z)f\|^2 &= L\left((|x-a|^2 + |b|^2)^2 |p|^2\right) \\ &= L\left(|x-a|^2(|x-a|^2 + |b|^2)|p|^2\right) + L\left(|b|^2(|x-a|^2 + |b|^2)|p|^2\right) \\ &= L\left(|x-a|^2(|x-a|^2 + |b|^2)|p|^2\right) + |b|^2\|f\|^2. \end{aligned}$$

The term on the left is nonnegative since the argument is a sum of squares, and L is a positive functional, thus for $f \in \mathcal{E}_z'$, $\|(X - z)f\| \geq |\Im z|\|f\|$. Since $\overline{X}$ is the closure of X, it follows that for any $\phi \in \mathcal{E}_z$,

$$\|(\overline{X} - z)\phi\| \geq |\Im z|\|\phi\|.$$

The above inequality implies that the range of $\overline{X} - z$ restricted to $\mathcal{E}_z$ is closed, thus the range of $\overline{X} - z$ is closed. □

For any $y = (y_1, \dots, y_d) \in \mathbb{C}^d$, we define the associated operator $y : \mathcal{H}^d \to \mathcal{H}$ by $yf = y_1 f_1 + \dots + y_d f_d$, where $f = (f_1, \dots, f_d)$. We estimate the norm

$$\|yf\| \leq \sum_{k=1}^{d} |y_k| \, \|f_k\| \leq \sum_{k=1}^{d} |y_k| \, \|f\|,$$

thus in the operator norm, $\|y\| \leq \sum_k |y_k|$.

In the following, we make use of the inequality on d-tuples of complex numbers

$$\sum_k |y_k| \leq \sqrt{d} \left(\sum_k |y_k|^2 \right)^{\frac{1}{2}} = \sqrt{d}|y|$$

by stating that if $|w - z| < \frac{|\Im z|}{\sqrt{d}}$, then $\|w - z\| < |\Im z|$.

Theorem 1. *The value of* $\dim \ker((\overline{X} - z)^*)$ *is constant in* $\mathbb{C}^d \backslash \mathbb{R}^d$.

Proof. We slightly modify the standard argument which shows that the defect indices of a closed symmetric operator are constant in the upper half and lower half-planes. Let $z \in \mathbb{C}^d \backslash \mathbb{R}^d$, and let $w \in \mathbb{C}^d$ so that $|z - w| < \frac{|\Im z|}{\sqrt{d}}$. We first prove that $\dim \ker((\overline{X} - z)^*) \geq \dim \ker((\overline{X} - w)^*)$.

Suppose then that $\dim \ker((\overline{X} - z)^*) < \dim \ker((\overline{X} - w)^*)$. Then there is some $u \in \ker((\overline{X} - z)^*)^\perp$ with $\|u\| = 1$ which is also contained in $\ker((\overline{X} - w)^*)$. Since $\ker((\overline{X} - z)^*)^\perp = \operatorname{ran}(\overline{X} - z)$, there is some $\phi \in \mathcal{E}_z$ so that $(\overline{X} - z)\phi = u$. Then

$$0 = |\langle (\overline{X} - w)^*, \phi \rangle| = |\langle u, (\overline{X} - z)\phi \rangle + \langle u, (z - w)\phi \rangle| \geq \|u\|^2 - \|z - w\|\|u\|\|\phi\|.$$

By the selection of w, $\|z-w\| < |\Im z|$, and $\|\phi\| \leq |\Im z| \|u\|$ since $\overline{X} - z$ is bounded from below on $\mathcal{E}_z$. Putting this into the above relations gives $0 > 0$. From this contradiction, we conclude that $\dim \ker((\overline{X} - z)^*) \leq \dim \ker((\overline{X} - w)^*)$.

Now by picking w so that $|w - z| < \frac{|\Im z|}{2\sqrt{d}}$, we repeat the above argument with the roles of z and w reversed, obtaining $\dim \ker((\overline{X} - w)^*) \leq \dim \ker((\overline{X} - z)^*)$. Thus for each $z \in \mathbb{C}^d \backslash \mathbb{R}^d$, there is some neighborhood U of z so that $\dim \ker((\overline{X} - w)^*)$ is constant on U. Since $\mathbb{C}^d \backslash \mathbb{R}^d$ is path connected, a compactness argument shows that $\dim \ker((\overline{X} - z)^*)$ is constant in this domain. □

We have used $\mathcal{E}_z$ as a set of representatives for the whole of the domain of $\overline{X} - z$ and $\mathcal{E}_w$ for $\overline{X} - w$. It is helpful to note that if z and w are close enough, then we may use a single effective domain.

Proposition 1. *Let $z \in \mathbb{C}^d \backslash \mathbb{R}^d$ and $w \in \mathbb{C}^d$ so that $|w - z| < \frac{|\Im z|}{\sqrt{d}}$. Then $\mathcal{E}_z$ is an effective domain of $\overline{X} - w$.*

Proof. From the discussion after Lemma 1, we first conclude that $\overline{X} - w$ restricted to the subspace $\mathcal{E}_z$ has closed range. For in this case, $\|w - z\| < |\Im z|$. Let $f \in \mathcal{E}_z$; then

$$\begin{aligned} \|(\overline{X} - w)f\| &\geq \|(\overline{X} - z)f\| - \|(z - w)f\| \\ &\geq (|\Im z| - \|w - z\|)\|f\|. \end{aligned}$$

Since $\|w - z\| < |\Im z|$, this means that $\overline{X} - w$ is bounded below on $\mathcal{E}_z$ and thus has closed range.

Now for the sake of argument, suppose that $(\overline{X} - w)\mathcal{E}_z$ is a proper subspace of $\operatorname{ran}(\overline{X} - w)$. We follow a similar argument as the above theorem. Since $\dim \ker((\overline{X} - z)^*) = \dim \ker((\overline{X} - w)^*)$, there is some $v \in \operatorname{ran}(\overline{X} - z)$ with $\|v\| = 1$ so that v is orthogonal to $(\overline{X} - w)\mathcal{E}_z$. Then there is $\phi \in \mathcal{E}_z$ so that $(\overline{X} - z)\phi = v$, and $\|\phi\| \leq |\Im z|$. Thus for every $\psi \in \mathcal{E}_z$,

$$\langle (\overline{X} - z)\phi, (\overline{X} - w)\psi \rangle = 0.$$

Setting $\psi = \phi$, we obtain $\|v\|^2 + \langle v, (z-w)\phi \rangle = 0$, which contradicts $\|z - w\| < |\Im z|$. Since $(\overline{X} - w)\mathcal{E}_z$ is closed, this implies that $(\overline{X} - w)\mathcal{E}_z = \operatorname{ran}(\overline{X} - w)$. Since $\overline{X} - w$ is bounded below, it is injective on $\mathcal{E}_z$, so this is an effective domain for $\overline{X} - w$. □

4. Reproducing kernels

For the operator X with cyclic vector 1, we assert that the defect index is either 0 or 1, since for any $z \in \mathbb{C}^d$, $\mathbb{C} \cdot 1 + (X - z)\mathbb{C}[x] = \mathbb{C}[x]$. If the defect index is 0, then $1 \in \operatorname{ran}(\overline{X} - z)$. If the defect index is 1, then we conclude that $1 \notin \operatorname{ran}(\overline{X} - z)$, since if it were, then $\operatorname{ran}(\overline{X} - z)$ would include the closure of the polynomials, and thus the range would be all of $\mathcal{H}$. Define $\rho(z)$ to be the square of the distance from 1 to the range of $\overline{X} - z$. Then for any $z \in \mathbb{C}^d \backslash \mathbb{R}^d$, $\rho(z) = 0$ if and only if the defect index is 0.

Assume that the defect index is 1. Then there is a unique $k_z \in \ker((\overline{X}-z)^*)$ so that $\langle k_z, 1\rangle = 1$. This k_z has the reproducing property: for any $p(x) \in \mathbb{C}[x]$, $\langle p(x), k_z\rangle = p(z)$. For each X_j,

$$\langle X_j 1, k_z\rangle = \langle x_j - z_j, k_z\rangle + \langle z_j, k_z\rangle = z_j,$$

and for each X_j and X_k,

$$\langle X_j X_k 1, k_z\rangle = \langle (x_j - z_j)x_k, k_z\rangle + \langle z_j x_k, k_z\rangle = z_j z_k.$$

We proceed inductively to achieve $\langle p(x), k_z\rangle = p(z)$.

We consider the standard orthonormal polynomials so that $P_{(0,\ldots,0)}(x) = 1$, and $\deg P_\alpha(x) = |\alpha|$. These are constructed by applying the Gram-Schmidt process to the monomials listed in order of nondecreasing degree; see [4] for more details. For our purposes, any complete orthonormal system of polynomials which contains the polynomial $p(x) = 1$ will suffice. Using $\langle P_\alpha(x), k_z\rangle = P_\alpha(z)$, we obtain the decomposition

$$k_z(x) = \sum_{\alpha\in\mathbb{N}_0^d} \overline{P_\alpha(z)} P_\alpha(x), \quad \text{from which follows} \quad \|k_z\|^2 = \sum_{\alpha\in\mathbb{N}_0^d} |P_\alpha(z)|^2.$$

If we decompose the vector $p(x) = 1$ with respect to the complementary subspaces $\operatorname{ran}(\overline{X}-z)$ and $\mathbb{C}\cdot k_z$, then the projection onto $\mathbb{C}\cdot k_z$ will have norm equal to the distance from 1 to $\operatorname{ran}(\overline{X}-z)$. The projection of the polynomial $p(x) = 1$ onto $\mathbb{C}\cdot k_z$ is $\frac{\langle 1, k_z\rangle}{\|k_z\|^2} k_z$. The norm of this vector is $\frac{1}{\|k_z\|}$, hence $\rho(z) = \frac{1}{\|k_z\|^2}$. In the case $d = 1$, this is consistent with the definition of $\rho(z)$ as given in [8] and [1].

The continuity of the polynomials gives the next result.

Theorem 2. *The function $z \mapsto k_z$ on $\mathbb{C}^d\backslash\mathbb{R}^d \to \mathcal{H}$ is weakly continuous.*

Proof. First assume that $\|k_w\|$ is uniformly bounded for all w in some neighborhood of z. Since each $p(x) \in \mathbb{C}[x]$ is continuous, then as $w \to z$, $p(w) \to p(z)$, or in other words $\langle p(x), k_w\rangle \to \langle p(x), k_z\rangle$. Since $k_w \to k_z$ weakly with respect to a dense subset of $\mathcal{H}$ and that $\|k_w\|$ is bounded for all w in a neighborhood of z, this implies that $\langle f, k_w\rangle \to \langle f, k_z\rangle$ for every $f \in \mathcal{H}$.

Now we show that $\|k_z\|$ is bounded in compact subsets of $\mathbb{C}^d\backslash\mathbb{R}^d$. For sake of contradiction, assume not, so there exists $z \in \mathbb{C}^d\backslash\mathbb{R}^d$ with $w_n \to z$ and $\|k_{w_n}\| > n$. Without loss of generality, we may assume that $|z - w_n| < \frac{|\Im z|}{2\sqrt{d}}$ for every $n \in \mathbb{N}$. Since $\|k_{w_n}\| > n$, the distance between 1 and $\operatorname{ran}(\overline{X} - w_n)$ is less than $\frac{1}{n}$, and let v_n be the projection of the element 1 onto $\operatorname{ran}(\overline{X} - w_n)$. Note that $\|v_n\| < 1$ since $\|1\| = 1$. Since $\mathcal{E}_z$ is an effective domain for $\overline{X} - w_n$, there is some element $f_n \in \mathcal{E}_z$ so that $(\overline{X} - w_n)f_n = v_n$. Since $\overline{X} - w_n$ is bounded below by $\frac{|\Im z|}{2\sqrt{d}}$ on $\mathcal{E}_z$, this implies that $\|f_n\| < \frac{2\sqrt{d}}{|\Im z|}$. Note that this implies that

$$\|(w_n - z)f_n\| \le \sqrt{d}|w_n - z|\|f_n\| < \frac{2d|w_n - z|}{|\Im z|}.$$

Pick $N \in \mathbb{N}$ so that $n \geq N$ implies $|z - w_n| < \frac{\sqrt{\rho(z)}|\Im z|}{4d}$. Let $n \in \mathbb{N}$ so that $n > \max(N, \frac{2}{\sqrt{\rho(z)}})$. Since $n > \frac{2}{\sqrt{\rho(z)}}$, we have

$$\|1 - (\overline{X} - w_n)f_n\| < \frac{\sqrt{\rho(z)}}{2}$$

as well as

$$\begin{aligned}\|1 - (\overline{X} - w_n)f_n\| &\geq \|1 - (\overline{X} - z)f_n\| - \|(z - w_n)f_n\| \\ &> \sqrt{\rho(z)} - \frac{2d|z - w_n|}{|\Im z|} > \sqrt{\rho(z)} - \frac{\sqrt{\rho(z)}}{2}.\end{aligned}$$

This contradiction implies that we cannot pick a sequence $w_n \to z$ so that $\|k_{w_n}\|$ is unbounded. Therefore, $\|k_w\|$ is bounded everywhere for all w within some neighborhood of z. □

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Commutative Algebras of Toeplitz Operators on the Super Upper Half-plane: Super Parabolic Case

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Abstract. We prove that there exist a unitary operator between the super weighted super Bergman spaces of the super-disk and the super upper plane, and we find the form of the functions invariant under the action of super reals over the super upper plane. We prove that, generalizing the parabolic classical case, every super Toeplitz operator with super-reals-invariant symbol is diagonal. Finally we prove that the algebra of Toeplitz operators with symbols invariant under the action of the super reals is commutative.

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1. Introduction

In [6] Grudski, Quiroga and Vasilevski showed that the C^*-algebra generated by the Toeplitz operators is commutative on each weighted Bergman space if and only if there is a pencil of hyperbolic geodesics such that the symbols of the Toeplitz operators are constant on the cycles of such a pencil. All cycles are, in fact, the orbits of a one-parameter subgroup of isometries for the hyperbolic geometry on the unit-disk. This provides us with the following scheme: the C^*-algebra generated by Toeplitz operators is commutative on each weighted Bergman space if and only if there is a maximal commutative subgroup of Möbius transformations such that the symbols of the Toeplitz operators are invariant under the action of this subgroup.

Others similar results on the sphere, ball, Reihart domains can be found in [11, 12, 13, 14].

In [2, 3] Borthwick, Klimek, Lesniewski and Rinaldi introduced a general theory of the non-perturbative quantization of a class of hermitian symmetric super-manifolds. The quantization scheme is based on the notion of a Toeplitz super-operator on a suitable $\mathbb{Z}_2$-graded Hilbert space of super-holomorphic functions. The quantized super-manifold arises as the $\mathbb{C}^*$-algebra generated by such operators. The authors made the quantization on the super-plane, super-disk, and Cartan super-domains.

In [10] we study commutative algebras of Toeplitz operators on the super-disk and we analyze the generalization corresponding to a classical elliptic case (symbols are invariant under the action of the circle), this generalization consists in two cases: symbols invariant under the action of the super circle.

The aim of this article is to continue the study of commutative algebras of super Toeplitz operators, now we work on the super upper plane and we analyze the generalization corresponding to a classical parabolic case (symbols are invariant under the action of the reals), this generalization consists in taking the symbols invariant under the action of super reals.

This article is organized as follows. In Section 2, we present some results about Toepliz operators over the super-disk. In Section 3, we give a unitary operator between the Bergman space of the super-disk and the respective Bergman space of the super upper plane. Thus, we find the form of the Bergman kernel on the super upper plane and we give the form of the Toeplitz operators on the super upper plane, which are unitarily equivalent to the Toeplitz operators on super-disk. In Section 4, we find the explicit form of the functions invariant under the action of super reals. Finally, in Section 5, we prove that every Toeplitz operator with super real invariant symbol is equivalent to multiplication operator. Therefore the C^* algebra generated by this operators is commutative.

2. Toeplitz operators on the super-disk

We present here the main results for the unit-disk, for more details we refer to [9]. Let $\mathcal{O}(\mathbb{B})$ denote the algebra of all holomorphic functions $\psi(z)$ on the open unit-disk

$$\mathbb{B} := \{z \in \mathbb{C} : \ |z| < 1\}.$$

Let Λ_1 denote the complex Grassmann algebra with generator ζ, satisfying the relation $\zeta^2 = 0$. Thus

$$\Lambda_1 = \mathbb{C}\langle 1, \zeta\rangle.$$

The tensor product algebra

$$\mathcal{O}(\mathbb{B}^{1|1}) := \mathcal{O}(\mathbb{B}) \otimes \Lambda_1 = \mathcal{O}(\mathbb{B})\langle 1, \zeta\rangle$$

consists of all "super-holomorphic" functions

$$\Psi = \psi_0 + \zeta\psi_1$$

with $\psi_0, \psi_1 \in \mathcal{O}(\mathbb{B})$. We sometimes write

$$\Psi(z, \zeta) = \psi_0(z) + \zeta\psi_1(z)$$

for all $z \in \mathbb{B}$.

Definition 2.1. For $\nu > 1$, the *weighted Bergman space*

$$H^2_\nu(\mathbb{B}) := \mathcal{O}(\mathbb{B}) \cap L^2(\mathbb{B}, d\mu_\nu)$$

consists of all holomorphic functions on $\mathbb{B}$ which are square-integrable for the probability measure

$$d\mu_\nu(z) = \frac{\nu - 1}{\pi}(1 - |z|^2)^{\nu-2} dz. \tag{2.1}$$

Where dz denotes the Lebesgue measure on $\mathbb{C}$.

It is well known [8] that $H^2_\nu(\mathbb{B})$ has the reproducing kernel

$$K_\nu(z, w) = (1 - z\overline{w})^{-\nu}$$

for all $z, w \in \mathbb{B}$. Let $\Lambda_1^{\mathbb{C}}$ denote the complex Grassmann algebra with 2 generators $\zeta, \overline{\zeta}$ satisfying

$$\zeta^2 = \overline{\zeta}^2 = 0, \quad \zeta\overline{\zeta} = -\overline{\zeta}\zeta.$$

Thus,

$$\Lambda_1^{\mathbb{C}} = \mathbb{C}\langle 1, \zeta, \overline{\zeta}, \overline{\zeta}\,\zeta\rangle = \Lambda_1\langle 1, \overline{\zeta}\rangle.$$

Let $\mathcal{C}(\overline{\mathbb{B}})$ denote the algebra of continuous functions on $\overline{\mathbb{B}}$. The tensor product

$$\mathcal{C}(\overline{\mathbb{B}}^{1|1}) := \mathcal{C}(\overline{\mathbb{B}}) \otimes \Lambda_1^{\mathbb{C}} = \mathcal{C}(\overline{\mathbb{B}})\langle 1, \zeta, \overline{\zeta}, \overline{\zeta}\zeta\rangle$$

consists of all "continuous super-functions"

$$F = f_{00} + \overline{\zeta}\, f_{10} + \zeta\, f_{01} + \overline{\zeta}\zeta\, f_{11}\,, \tag{2.2}$$

where $f_{00}, f_{10}, f_{01}, f_{11} \in \mathcal{C}(\overline{\mathbb{B}})$. The involution on $\mathcal{C}(\overline{\mathbb{B}}^{1|1})$ is given by

$$\overline{F} = \overline{f}_{00} + \zeta\,\overline{f}_{10} + \overline{\zeta}\,\overline{f}_{01} + \overline{\zeta}\zeta\,\overline{f}_{11}$$

where $\overline{f}(z) := \overline{f(z)}$ (pointwise conjugation).

The algebra $\mathcal{C}(\overline{\mathbb{B}}^{1|1})$ contains $\mathcal{O}(\mathbb{B}^{1|1})$ as a subalgebra, and for $\Psi = \psi_0 + \zeta\,\psi_1 \in \mathcal{O}(\mathbb{B}^{1|1})$ we have

$$\overline{\Psi}\Psi = \overline{\psi}_0\psi_0 + \zeta\,\overline{\psi}_0\,\psi_1 + \overline{\zeta}\,\overline{\psi}_1\,\psi_0 + \overline{\zeta}\zeta\,\overline{\psi}_1\,\psi_1.$$

Given a super-function $F \in \mathcal{C}(\overline{\mathbb{B}}^{1|1})$, we define its *Berezin integral*

$$\int\limits_{\mathbb{C}^{0|1}} d\zeta\ F := f_{11} \in \mathcal{C}(\overline{\mathbb{B}})$$

and

$$\int\limits_{\mathbb{B}^{1|1}} dz\, d\zeta\ F(z, \zeta) := \int\limits_{\mathbb{B}} dz \int\limits_{\mathbb{C}^{0|1}} d\zeta\ F(z, \zeta) = \int\limits_{\mathbb{B}} dz\ f_{11}(z). \tag{2.3}$$

Thus the "fermionic integration" is determined by the rules

$$\int\limits_{\mathbb{C}^{0|1}} d\zeta \cdot \zeta = \int\limits_{\mathbb{C}^{0|1}} d\zeta \cdot \overline{\zeta} = \int\limits_{\mathbb{C}^{0|1}} d\zeta \cdot 1 = 0\,, \quad \int\limits_{\mathbb{C}^{0|1}} d\zeta \cdot \overline{\zeta}\zeta = 1.$$

As an example, we have

$$\int_{\mathbb{B}^{1|1}} dz\, d\zeta\ \overline{F}(z,\zeta)\, F(z,\zeta)$$
$$= \int_{\mathbb{B}} dz(\overline{f_{00}(z)}\, f_{11}(z) + \overline{f_{11}(z)}\, f_{00}(z) - \overline{f_{10}(z)}\, f_{10}(z) + \overline{f_{01}(z)}\, f_{01}(z)),$$

which shows that the (unweighted) Berezin integral is not positive. For $\Psi = \psi_0 + \zeta\psi_1 \in \mathcal{O}(\mathbb{B}^{1|1})$, we see that

$$\int_{\mathbb{B}^{1|1}} dz\, d\zeta\ \overline{\Psi}(z,\zeta)\, \Psi(z,\zeta) = \int_{\mathbb{B}^1} dz\ \overline{\psi_1(z)}\psi_1(z)$$

is positive, but not positive definite since the ψ_0 term is not present.

Definition 2.2. For any parameter $\nu > 1$ the (weighted) *super-Bergman space*

$$H^2_\nu(\mathbb{B}^{1|1}) \subset \mathcal{O}(\mathbb{B}^{1|1})$$

consists of all super-holomorphic functions $\Psi(z,\zeta)$ which satisfy the square-integrability condition

$$(\Psi|\Psi)_\nu := \frac{1}{\pi}\int_{\mathbb{B}^{1|1}} dz\, d\zeta\ (1 - z\overline{z} - \zeta\overline{\zeta})^{\nu-1}\, \overline{\Psi(z,\zeta)}\, \Psi(z,\zeta) < +\infty.$$

Proposition 2.3. *For $\Psi = \psi_0 + \zeta\,\psi_1 \in \mathcal{O}(\mathbb{B}^{1|1})$ we have*

$$\frac{1}{\pi}\int_{\mathbb{B}^{1|1}} dz\, d\zeta\ (1 - z\overline{z} - \zeta\overline{\zeta})^{\nu-1}\overline{\Psi}(z,\zeta)\, \Psi(z,\zeta) = (\psi_0|\psi_0)_\nu + \frac{1}{\nu}\,(\psi_1|\psi_1)_{\nu+1}\,,$$

i.e., there exists an orthogonal decomposition

$$H^2_\nu(\mathbb{B}^{1|1}) = H^2_\nu(\mathbb{B}) \oplus [H^2_{\nu+1}(\mathbb{B}) \otimes \Lambda^1(\mathbb{C}^1)]$$

into a sum of weighted Bergman spaces, where $\Lambda^1(\mathbb{C}^1)$ is the one-dimensional vector space with basis vector ζ.

Proposition 2.4. *For $\Psi = \psi_0 + \zeta\,\psi_1 \in H^2_\nu(\mathbb{B}^{1|1})$ we have the reproducing kernel property*

$$\Psi(z,\zeta) = \frac{1}{\pi}\int_{\mathbb{B}^{1|1}} dw\, d\omega\ (1 - w\overline{w} - \omega\overline{\omega})^{\nu-1}(1 - z\overline{w} - \zeta\overline{\omega})^{-\nu}\Psi(w,\omega),$$

i.e., $H^2_\nu(\mathbb{B}^{1|1})$ has the reproducing kernel

$$K_\nu(z,\zeta,w,\omega) = (1 - z\overline{w} - \zeta\overline{\omega})^{-\nu}.$$

For $F \in \mathcal{C}(\overline{\mathbb{B}}^{1|1})$, the *super-Toeplitz operator* $T^{(\nu)}_F$ on $H^2_\nu(\mathbb{B}^{1|1})$ is defined as

$$T^{(\nu)}_F\Psi = P^{(\nu)}(F\Psi),$$

where $P^{(\nu)}$ denotes the orthogonal projection onto $H^2_\nu(\mathbb{B}^{1|1})$.

Theorem 2.5. *With respect to the decomposition $\Psi = \psi_0 + \zeta\,\psi_1$, the super-Toeplitz operator $T_F^{(\nu)}$ on $H_\nu^2(\mathbb{B}^{1|1})$ is given by the block matrix*

$$T_F^{(\nu)} = \begin{pmatrix} T_\nu^\nu\left(f_{00} + \frac{1-w\overline{w}}{\nu-1}\, f_{11}\right) & T_\nu^{\nu+1}\left(\frac{1-w\overline{w}}{\nu-1}\, f_{10}\right) \\ T_{\nu+1}^\nu\,(f_{01}) & T_{\nu+1}^{\nu+1}\,(f_{00}) \end{pmatrix}. \tag{2.4}$$

Here $T_{\nu+i}^{\nu+j}(f)$, for $0 \leq i, j \leq 1$, denotes the Toeplitz type operator from $H_{\nu+j}^2(\mathbb{B})$ to $H_{\nu+i}^2(\mathbb{B})$ defined by

$$T_{\nu+i}^{\nu+j}(f)\,\psi := P_{\nu+i}(f\psi)$$

for $\psi \in H_{\nu+j}^2(\mathbb{B})$ and $P_{\nu+i}$ is the orthogonal projection from $L_{\nu+i}^2(\mathbb{B})$ onto $H_{\nu+i}^2(\mathbb{B})$.

3. Toeplitz operators on the super upper half-plane

In this section we present the relationship between the super-disk and the super-plane, and its respective Bergman space.

The super upper half-plane $\overline{\mathbb{H}}^{1|1}$ is the supermanifold $(H, \mathcal{O})$, where $\mathbb{H} = \{z \in \mathbb{C} :\ \mathrm{Im} z > 0\}$, and where $\mathcal{O}$ is the sheaf of superalgebras on H whose space of global sections is $C^\infty(\overline{\mathbb{H}}^{1|1}) = C^\infty(H) \otimes \bigwedge(\mathbb{C})$ where $\bigwedge(\mathbb{C})$ denotes the exterior algebra over $\mathbb{C} = \mathbb{R}^2$. We denotes the standard generators of $\bigwedge(\mathbb{C})$ by η and $\bar{\eta}$. Thus, an element $f \in C^\infty(\overline{\mathbb{H}}^{1|1})$ can be written as

$$f(z, \zeta, \bar{\zeta}) = f_{00}(z) + f_{00}(z)\zeta + f_{00}(z)\bar{\zeta} + f_{00}(z)\bar{\zeta}\zeta$$

where $f_{ij} \in C^\infty(H)$.

Now we see that there exists a diffeomorphism of supermanifolds between the super-disk and super upper half-plane.

We define the super matrix

$$\psi = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\mathrm{Ber}(\psi) = 1$.

Moreover, the inverse matrix is given by

$$\psi^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix ψ induces a morphism from the super-disk $\overline{\mathbb{B}}^{1|1}$ to the super upper half-plane $\overline{\mathbb{H}}^{1|1}$ defined by $\psi(z, \zeta) = (w, \eta)$ where

$$w = \frac{z + i}{iz + 1} \quad \text{and} \quad \eta = \frac{\zeta\sqrt{2}}{iz + 1}. \tag{3.1}$$

Similarly, the inverse matrix induces the inverse function given by $\psi^{-1}(w,\eta)=(z,\theta)$ where

$$z=\frac{w-i}{-iw+1} \quad \text{and} \quad \zeta=\frac{\eta\sqrt{2}}{-iw+1}. \tag{3.2}$$

The Lie super-group $SL_{(2|2)}(\mathbb{R})$ is defined as follows, its base manifold is $SL_2(\mathbb{R})$ and its structure sheaf is generated by γ_{ij} and $\bar{\gamma}_{ij}$ for $1\le i,j\le 3$, with the following parity assignments:

$$|\gamma_{ij}|=|\bar{\gamma}_{ij}|=\begin{cases} 0, & \text{if } 1\le i,j\le 2 \text{ and } i=j=3,\\ 1, & \text{otherwise.}\end{cases} \tag{3.3}$$

This means that, if $|\gamma_{ij}|=0$ then γ_{ij} is an even super-number, in other case γ_{ij} is odd.

Let $\gamma=\{\gamma_{ij}\}$ denote the super-matrix with entries γ_{ij} and let γ^* be its hermitian adjoint, where $\gamma^*_{ij}=\bar{\gamma}_{ji}$.

For $\gamma\in SL_{(2|2)}(\mathbb{R})$ we assume that

$$\gamma^* I\gamma=I, \tag{3.4}$$

where

$$I=\begin{pmatrix} 0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & -1\end{pmatrix}, \tag{3.5}$$

and that

$$\mathrm{Ber}\gamma=1, \tag{3.6}$$

where Ber denotes the Berezinian (see, [1]).

The above conditions are the relations defining the structure sheaf of $SL_{(2|2)}(\mathbb{R})$. Multiplication is defined in the obvious way. We defined an action of $SL_{(2|2)}(\mathbb{R})$ on $\overline{\mathbb{H}}^{1|1}$ as follows

$$\begin{aligned} z\to z' &:= \frac{\gamma_{11}z+\gamma_{12}+\gamma_{13}\theta}{\gamma_{21}z+\gamma_{22}+\gamma_{23}\theta}\ ,\\ \theta\to\theta' &:= \frac{\gamma_{31}z+\gamma_{32}+\gamma_{33}\theta}{\gamma_{21}z+\gamma_{22}+\gamma_{23}\theta}\ . \end{aligned} \tag{3.7}$$

The expression $(\gamma_{21}z+\gamma_{22}+\gamma_{23}\theta)^{-1}$ is defined in terms of the Taylor series for super-functions (see, [1]) by

$$(\gamma_{21}z+\gamma_{22}+\gamma_{23}\theta)^{-1}=\frac{1}{\gamma_{21}z+\gamma_{22}}-\frac{\gamma_{23}}{(\gamma_{21}z+\gamma_{22})^2}\theta.$$

By a slight abuse of notation, we write (3.7) as $Z'=(z',\theta')=\gamma(Z)$.

Remark 3.1. The super group $SU(1,1|1)$ is defined in [2] and we can even prove that $SL_{(2|2)}(\mathbb{R})$ is isomorphic to $SU(1,1|1)$, where the isomorphism is given by

$$\gamma\longrightarrow\psi\gamma\psi^{-1}. \tag{3.8}$$

Let $\gamma \in SU(1,1|1)$ such that $\gamma^* J\gamma = J$, by the above map $\beta = \psi\gamma\psi^{-1}$ or $\gamma = \psi^{-1}\beta\psi$, then

$$(\psi^{-1}\beta\psi)^* J\psi^{-1}\beta\psi = J.$$

Equivalently

$$\beta^*\psi J\psi^{-1}\beta = \psi J\psi^{-1},$$

where

$$I = \psi \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \psi^{-1} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore, we have that $\beta \in SL_{(2|2)}(\mathbb{R})$ and the morphism given by (3.8) is isomorphism.

If there exists a morphism γ between super domains then we defined

$$\gamma'(Z) = \mathrm{Ber} \begin{pmatrix} \frac{\partial z'}{\partial z} & \frac{\partial \theta'}{\partial z} \\ \frac{\partial z'}{\partial \theta} & \frac{\partial \theta'}{\partial \theta} \end{pmatrix} = \mathrm{Ber}\frac{\partial Z'}{\partial Z}. \tag{3.9}$$

for more details see [1].

Lemma 3.2. *Let* $Z_1 = (z_1, \zeta)$, $z_2 = (z_2, \zeta_2)$, $\psi(Z_1) = (w_1, \eta_1) = W_1$ *and* $\psi(Z_2) = (w_2, \eta_2) = W_2$ *where* ψ *is given by* (3.1) *then,*

$$\frac{w_1 - \bar{w}_2}{i} - \eta_1\bar{\eta}_2 = (1 + z_1\bar{z}_2 - \zeta_1\bar{\zeta}_2)\psi'(Z_1)\overline{\psi'(Z_2)}, \tag{3.10}$$

$$(1 + z_1\bar{z}_2 - \zeta_1\bar{\zeta}_2) = (\frac{w_1 - \bar{w}_2}{i} - \eta_1\bar{\eta}_2)(\psi^{-1})'(W_1)\overline{(\psi^{-1})'(W_2)}. \tag{3.11}$$

Proof. We calculate the Berezian of the Jacobian matrix

$$\psi'(Z_j) = \mathrm{Ber} \begin{pmatrix} 2(iz_j + 1)^{-2} & -\sqrt{2}i\zeta_j(iz_j + 1)^{-2} \\ 0 & \sqrt{2}(iz_j + 1)^{-1} \end{pmatrix} = \frac{\sqrt{2}}{iz_j + 1}$$

where $j = 1, 2$.

Using the above and substituting $\psi(Z_j) = (w_j, \eta_j)$ on the left-hand side of equation (3.10) as follows

$$\begin{aligned}
\frac{w_1 - \bar{w}_2}{i} - \eta_1\bar{\eta}_2 &= \frac{1}{i}\left(\frac{z_1 + i}{iz_1 + 1} - \overline{\left(\frac{z_2 + i}{iz_2 + 1}\right)}\right) - \frac{\zeta_1\sqrt{2}}{iz_1 + 1}\overline{\left(\frac{\zeta_2\sqrt{2}}{iz_2 + 1}\right)} \\
&= \frac{1}{(iz_1 + 1)(-i\bar{z}_2 + 1)}\left(\frac{1}{i}\left((z_1 + i)(-i\bar{z}_2 + 1) - (\bar{z}_2 - i)(iz_1 + 1)\right) - 2\zeta_1\bar{\zeta}_2\right) \\
&= \frac{1}{(iz_1 + 1)(-iz_2 + 1)}\left(\frac{1}{i}(-i\bar{z}_2 z_1 + z_1 + \bar{z}_2 + i - i\bar{z}_2 z_1 - z_1 - \bar{z}_2 + i) - 2\zeta_1\bar{\zeta}_2\right) \\
&= \frac{1}{(iz_1 + 1)(-iz_2 + 1)}(2 - 2z_1\bar{z}_2 - 2\zeta_1\bar{\zeta}_2) = (1 - z_1\bar{z}_2 - \zeta_1\bar{\zeta}_2)\psi'(Z_1)\overline{\psi'(Z_2)}.
\end{aligned}$$

Analogously by equation 3.11. □

In [2] it was proved that the measure invariant under the action of $SU(1,1|1)$ is

$$\frac{1}{\pi}(1 - z\bar{z} - \theta\bar{\theta})^{-1} dz d\bar{z} d\theta d\bar{\theta}.$$

By Lemma 3.2 and equation (3.8), we have that the invariant measure of the super-disk corresponds to the measure of the super upper half-plane as follows

$$\frac{1}{\pi}(1 - z\bar{z} - \theta\bar{\theta})^{-1} dz d\bar{z} d\theta d\bar{\theta} = \frac{1}{\pi}(2\mathrm{Im} w - \eta\bar{\eta})^{-1} dw d\bar{w} d\eta d\bar{\eta}$$

where $\psi(z, \theta) = (w, \eta)$. Moreover, the measure of the plane is invariant under the action of the group $SL_{(2|2)}(\mathbb{R})$.

Definition 3.3. For $\nu > 1$, the *weighted Bergman space*

$$H^2_\nu(\mathbb{H}) := \mathcal{O}(\mathbb{H}) \cap L^2(\mathbb{H}, d\omega_\nu)$$

consists of all holomorphic functions on $\mathbb{H}$ which are square-integrable for the probability measure.

$$d\omega_\nu(z) = \frac{\nu - 1}{\pi} (z - \bar{z})^{\nu - 2} dz. \tag{3.12}$$

where dz denotes Lebesgue measure on $\mathbb{C}$.

It is well known (see for example [16]) that $H^2_\nu(\mathbb{H})$ has the reproducing kernel

$$K_\nu(z, w) = (z - \bar{w})^{-\nu}$$

for all $z, w \in \mathbb{H}$.

Let $\mathcal{C}(\overline{\mathbb{H}})$ denote the algebra of continuous functions on $\overline{\mathbb{H}}$.

The tensor product

$$\mathcal{C}(\overline{\mathbb{H}}^{1|1}) := \mathcal{C}(\overline{\mathbb{H}}) \otimes \bigwedge \mathbb{C}$$

consists of all "continuous super-functions"

$$F = f_{00} + \bar{\zeta} f_{10} + \zeta f_{01} + \bar{\zeta}\zeta f_{11}, \tag{3.13}$$

where $f_{00}, f_{10}, f_{01}, f_{11} \in \mathcal{C}(\overline{\mathbb{H}})$.

Definition 3.4. For any parameter $\nu > 1$ the (weighted) *super-Bergman space*

$$H^2_\nu(\mathbb{H}^{1|1}) \subset \mathcal{O}(\mathbb{H}^{1|1})$$

consists of all super-holomorphic functions $\Psi(z, \zeta)$ that satisfy the square-integrability condition

$$(\Psi|\Psi)_\nu := \frac{1}{\pi} \int\limits_{\mathbb{H}^{1|1}} dz\, d\zeta\; (2\mathrm{Im}(z) - \zeta\bar{\zeta})^{\nu-1}\, \overline{\Psi(z, \zeta)}\, \Psi(z, \zeta) < +\infty.$$

Proposition 3.5. *For $\Psi = \psi_0 + \zeta\,\psi_1 \in \mathcal{O}(\mathbb{H}^{1|1})$ we have*

$$\frac{1}{\pi} \int\limits_{\mathbb{H}^{1|1}} dz\, d\zeta\; (2Im(z) - \zeta\bar{\zeta})^{\nu-1} \overline{\Psi}(z, \zeta)\, \Psi(z, \zeta) = (\psi_0|\psi_0)_\nu + \frac{1}{\nu}\, (\psi_1|\psi_1)_{\nu+1}\,,$$

i.e., there is an orthogonal decomposition

$$H^2_\nu(\mathbb{H}^{1|1}) = H^2_\nu(\mathbb{H}) \oplus [H^2_{\nu+1}(\mathbb{H}) \otimes \Lambda^1(\mathbb{C}^1)]$$

into a sum of weighted Bergman spaces, where $\Lambda^1(\mathbb{C}^1)$ *is the one-dimensional vector space with basis vector* ζ*.*

Let us now introduce the operator $U_\nu : H^2_\nu(\mathbb{B}^{1|1}) \longrightarrow H^2_\nu(\mathbb{H}^{1|1})$ by the rule

$$(U_\nu\Psi)(w,\omega) = \Psi\left(\frac{w-i}{1-iw}, \frac{\sqrt{2}\omega}{1-iw}\right)\left(\frac{\sqrt{2}}{1-iw}\right)^\nu$$

and its inverse $U_\nu^{-1} : H^2_\nu(\mathbb{H}^{1|1}) \longrightarrow H^2_\nu(\mathbb{B}^{1|1})$ which is given by

$$(U_\nu^{-1}\Psi)(z,\zeta) = \Psi\left(\frac{z+i}{1+iz}, \frac{\sqrt{2}\zeta}{1+iz}\right)\left(\frac{\sqrt{2}}{1+iz}\right)^\nu.$$

We check now that the operator U_ν is unitary,

$$\begin{aligned}
&\langle \Psi_1(z,\zeta), U_\nu^{-1}(\Psi_2)(z,\zeta)\rangle_{H^2_\nu(\mathbb{B}^{1|1})} \\
&= \frac{1}{\pi}\int_{\mathbb{B}^{1|1}} dz\, d\zeta\ (1 - z\overline{z} - \zeta\overline{\zeta})^{\nu-1}\, \overline{\Psi_1(z,\zeta)}\, \Psi_2\left(\frac{z+i}{1+iz}, \frac{\sqrt{2}\zeta}{1+iz}\right)\left(\frac{\sqrt{2}}{1+iz}\right)^\nu \\
&= \frac{1}{\pi}\int_{\mathbb{H}^{1|1}} dw\, d\eta\ \left(\frac{\sqrt{2}}{1-iw}\right)\left(\frac{\sqrt{2}}{1+i\bar{w}}\right)\left(\frac{w-\bar{w}}{i} - \eta\bar{\eta}\right)^{\nu-1} \\
&\quad \left(\frac{\sqrt{2}}{1-iw}\right)^{\nu-1}\overline{\left(\frac{\sqrt{2}}{1-iw}\right)^{\nu-1}\Psi_1\left(\frac{w-i}{1-iw}, \frac{\sqrt{2}\eta}{1-iw}\right)}\,\Psi_2\,(w,\eta)\left(\frac{1-iw}{\sqrt{2}}\right)^\nu \\
&= \frac{1}{\pi}\int_{\mathbb{H}^{1|1}} dw\, d\eta\ \left(\frac{w-\bar{w}}{i} - \eta\bar{\eta}\right)^{\nu-1}\overline{\left(\frac{\sqrt{2}}{1-iw}\right)^{\nu}\Psi_1\left(\frac{w-i}{1-iw}, \frac{\sqrt{2}\eta}{1-iw}\right)}\,\Psi_2\,(w,\eta) \\
&= \langle U_\nu(\Psi_1)(w,\eta), \Psi_2(w,\eta)\rangle_{H^2_\nu(\mathbb{H}^{1|1})}.
\end{aligned}$$

Proposition 3.6. *For* $\Psi = \psi_0 + \eta\,\psi_1 \in H^2_\nu(\mathbb{H}^{1|1})$ *we have the reproducing kernel property*

$$\Psi(w,\eta) = \frac{1}{\pi}\int_{\mathbb{H}^{1|1}} dx\, d\xi\ (2Im(x) - \xi\overline{\xi})^{\nu-1}\left(\frac{w-\bar{x}}{i} - \eta\overline{\xi}\right)^{-\nu}\Psi.(x,\xi),$$

i.e., $H^2_\nu(\mathbb{H}^{1|1})$ *has the reproducing kernel*

$$K_\nu(w,\eta,x,\xi) = \left(\frac{w-\bar{x}}{i} - \eta\overline{\xi}\right)^{-\nu}.$$

Proof. The Bergman projection $B_{\mathbb{H}^{1|1},\nu}$ clearly has the form

$$B_{\mathbb{H}^{1|1},\nu} = U_\nu B_{\mathbb{B}^{1|1},\nu} U_\nu^{-1}.$$

Note that $\psi(z,\zeta)=(w,\eta)$,$\psi(u,v)=(x,\xi)$ and $\psi^{-1}(w,\eta)=(z,\zeta)$, $\psi^{-1}(v,\xi)=(u,v)$ given by equations (3.1) and (3.2).

Calculate

$$\begin{aligned}
&U_\nu B_{\mathbb{B}^{1|1},\nu}U_\nu^{-1}(\Psi)(w,\eta)\\
&\quad=\left(\frac{\sqrt{2}}{1-iw}\right)^{\nu}\frac{1}{\pi}\int\limits_{\mathbb{B}^{1|1}} du\,dv\ (1-u\overline{u}-v\overline{v})^{\nu-1}(1-z\overline{u}-\zeta\overline{v})^{-\nu}\\
&\qquad\Psi\left(\frac{u+i}{1+iu},\frac{\sqrt{2}v}{1+iu}\right)\left(\frac{\sqrt{2}}{1+iu}\right)^{\nu}\\
&\quad=\frac{1}{\pi}\int\limits_{\mathbb{B}^{1|1}} dx\,d\xi\ (\mathrm{Im}(x)-\xi\overline{\xi})^{\nu-1}\left(\frac{w-\bar{x}}{i}-\eta\overline{\xi}\right)^{-\nu}\Psi\left(x,\xi\right). \qquad \square
\end{aligned}$$

For $F\in\mathcal{C}(\overline{\mathbb{H}}^{1|1})$, the *super-Toeplitz operator* $T_F^{(\nu)}$ on $H^2_\nu(\mathbb{H}^{1|1})$ is defined as

$$T_F^{(\nu)}\Psi=P^{(\nu)}(F\Psi),$$

where $P^{(\nu)}$ denotes the orthogonal projection onto $H^2_\nu(\mathbb{H}^{1|1})$.

Theorem 3.7. *With respect to the decomposition $\Psi=\psi_0+\zeta\,\psi_1$, the super-Toeplitz operator $T_F^{(\nu)}$ on $H^2_\nu(\mathbb{H}^{1|1})$ is given by the block matrix*

$$T_F^{(\nu)}=\begin{pmatrix} T_\nu^\nu\left(f_{00}+\frac{2Im(w)}{\nu-1}\ f_{11}\right) & T_\nu^{\nu+1}\left(\frac{2Im(w)}{\nu-1}\ f_{10}\right)\\ T_{\nu+1}^\nu\left(f_{01}\right) & T_{\nu+1}^{\nu+1}\left(f_{00}\right)\end{pmatrix}. \tag{3.14}$$

Here for $0\le i,j\le 1$, $T_{\nu+i}^{\nu+j}(f)$ denotes the Toeplitz type operator from $H^2_{\nu+j}(\mathbb{H})$ to $H^2_{\nu+i}(\mathbb{B})$ defined by

$$T_{\nu+i}^{\nu+j}(f)\,\psi:=P_{\nu+i}(f\psi),$$

for $\psi\in H^2_{\nu+j}(\mathbb{H})$ and $P_{\nu+i}$ is the orthogonal projection from $L^2_{\nu+i}(\mathbb{B})$ onto $H^2_{\nu+i}(\mathbb{H})$.

Proof. First we expand the follows expressions

$$\begin{aligned}
(2\mathrm{Im}(z)-\eta\overline{\eta})^{\nu-1}&=(2\mathrm{Im}(z))^{\nu-1}-(\nu-1)(2\mathrm{Im}(z))^{\nu-2}\zeta\overline{\zeta},\\
\left(\frac{w-\bar{z}}{i}-\eta\overline{\zeta}\right)^{-\nu}&=\left(\frac{w-\bar{z}}{i}\right)^{-\nu}+\nu\left(\frac{w-\bar{z}}{i}\right)^{-(\nu+1)}\eta\overline{\zeta}.
\end{aligned}$$

Now the Toeplitz operator with symbol F applied to Ψ is given by

$$\begin{aligned}
T_F^{(\nu)}(\psi_0(z)+\zeta\,\psi_1(z))(w,\eta)&=\frac{1}{\pi}\int\limits_{\mathbb{H}^{1|1}}(2\mathrm{Im}(z)-\zeta\overline{\zeta})^{\nu-1}\left(\frac{w-\bar{z}}{i}-\eta\overline{\zeta}\right)^{-\nu}\\
&\quad(f_{00}(z)+\overline{\zeta}\,f_{10}(z)+\zeta\,f_{01}(z)+\overline{\zeta}\zeta\,f_{11}(z))(\psi_0(z)+\zeta\,\psi_1(z))dz\,d\zeta.
\end{aligned}$$

Equivalently,

$$T_F^{(\nu)}(\Psi)(w,\eta) = \frac{\nu-1}{\pi}\int\limits_{\mathbb{H}} f_{00}(z)\psi_0(z)(2\mathrm{Im}(z))^{\nu-2}\left(\frac{w-\bar{z}}{i}\right)^{-\nu} dz \tag{3.15}$$
$$+\frac{1}{\pi}\int\limits_{\mathbb{H}} f_{11}(z)\psi_0(z)(2\mathrm{Im}(z))^{\nu-1}\left(\frac{w-\bar{z}}{i}\right)^{-\nu} dz$$
$$+\frac{1}{\pi}\int\limits_{\mathbb{H}} f_{10}(z)\psi_1(z)(2\mathrm{Im}(z))^{\nu-1}\left(\frac{w-\bar{z}}{i}\right)^{-\nu} dz$$
$$+\left(\frac{\nu}{\pi}\int\limits_{\mathbb{H}} f_{01}(z)\psi_0(z)(2\mathrm{Im}(z))^{\nu-1}\left(\frac{w-\bar{z}}{i}\right)^{-(\nu+1)} dz\right)\eta$$
$$+\left(\frac{\nu}{\pi}\int\limits_{\mathbb{H}} f_{00}(z)\psi_1(z)(2\mathrm{Im}(z))^{\nu-1}\left(\frac{w-\bar{z}}{i}\right)^{-(\nu+1)} dz\right)\eta.$$

Using the above equation and by how $T_{\nu+i}^{\nu+j}(f)$ is defined, we obtain the result. □

4. The super group $\mathbb{R}^{1|1}$

The super group $\mathbb{R}^{1|1}$ can be seen as the subgroup of the supergroup $SL_{(2|2)}(\mathbb{R})$ is defined by

$$M(h,\tau) = \begin{pmatrix} 1 & h & \tau \\ 0 & 1 & 0 \\ 0 & -i\tau & 1 \end{pmatrix}$$

where $\tau^* = \tau$ and $h \in \mathbb{R}$.

We show that $M(h,\tau) \in SL_{(2|2)}(\mathbb{R})$

$$M(h,\tau)^* I M(h,\tau) = \begin{pmatrix} 1 & 0 & 0 \\ h & 1 & i\tau \\ \tau & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & h & \tau \\ 0 & 1 & 0 \\ 0 & -i\tau & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We prove that the product of elements in $\mathbb{R}^{1|1}$ belong in $\mathbb{R}^{1|1}$, i.e.,

$$\begin{pmatrix} 1 & h_1 & \tau_1 \\ 0 & 1 & 0 \\ 0 & -i\tau_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & h_2 & \tau_2 \\ 0 & 1 & 0 \\ 0 & -i\tau_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & h_1+h_2 & \tau_1+\tau_2 \\ 0 & 1 & 0 \\ 0 & i(\tau_1+\tau_2) & 1 \end{pmatrix}.$$

Therefore the action of $\mathbb{R}^{1|1}$ on the $\mathbb{H}^{1|1}$ is given by $M(h,\tau)(z,\zeta) = (w,\eta)$ where

$$w = z + h + \tau\zeta \quad \text{and} \quad \eta = -i\tau + \zeta$$

Theorem 4.1. *Let f be a smooth function on the super upper half-plane. If f is invariant under the action of $\mathbb{R}^{1|1}$. Then f has the form*

$$f(z,\zeta) = f_0(y) + f_1(y)\zeta + f_1(y)\bar{\zeta} + \frac{f_0'(y)}{2}\bar{\zeta}\zeta. \tag{4.1}$$

Proof. A smooth function f on the super upper half-plane has the form

$$f(z,\zeta) = f_{00}(z) + f_{10}(z)\zeta + f_{01}(z)\bar{\zeta} + f_{11}(z)\bar{\zeta}\zeta,$$

where f_{ij} are smooth functions.

Now, we want to find the $\mathbb{R}^{1|1}$-invariant function on the super-plane, i.e.,

$$f(z,\zeta) = f(z+h+\tau\zeta, -i\tau+\zeta).$$

First we take the elements of the form $M(h,0)$, then f is invariant under the action of those elements if f satisfies the follows equation

$$\begin{aligned} &f_{00}(z) + f_{10}(z)\zeta + f_{01}(z)\bar{\zeta} + f_{11}(z)\bar{\zeta}\zeta \\ &\quad = f_{00}(z+h) + f_{10}(z+h)\zeta + f_{01}(z+h)\bar{\zeta} + f_{11}(z+h)\bar{\zeta}\zeta. \end{aligned}$$

By the above equation, we have that the functions f_{ij} depend on the y where $y=\text{Im}(z)$. Therefore, we obtain that f has the form

$$f(z,\zeta) = f_{00}(y) + f_{10}(y)\zeta + f_{01}(y)\bar{\zeta} + f_{11}(y)\bar{\zeta}\zeta \tag{4.2}$$

where y is the imaginary part of z.

Now, we consider the action of elements of the form $M(0,\tau)$, then

$$w = z + \tau\zeta \text{ and } \eta = -i\tau + \zeta$$

and we note that

$$\text{Im}(w) = \text{Im}(z) + \frac{\tau(\zeta+\bar{\zeta})}{2i}.$$

We take a function f that depends on $\text{Im}(z)$, then we define $h(\text{Im}(w))$ in term of Taylor series for super function (see [1]), thus

$$h(\text{Im}(w)) = h(y) + \frac{\tau(\zeta+\bar{\zeta})}{2i}h'(y),$$

where $y = \text{Im}(w)$. On the other hand

$$h(\text{Im}(w))\eta = (h(y) + \frac{\tau(\zeta+\bar{\zeta})}{2i}h'(y))(-i\tau+\zeta) = h(y)\zeta - ih(y)\tau.$$

Similarly,

$$h(\text{Im}(w))\bar{\eta} = (h(y) + \frac{\tau(\zeta+\bar{\zeta})}{2i}h'(y))(i\tau+\bar{\zeta}) = h(y)\bar{\zeta} + ih(y)\tau$$

and

$$h(\text{Im}(w))\eta\bar{\eta} = (h(y) + \frac{\tau(\zeta+\bar{\zeta})}{2i}h'(y))(i\tau+\bar{\zeta})(-i\tau+\zeta) = h(y)\bar{\zeta}\zeta + ih(y)\tau(\zeta+\bar{\zeta}).$$

As a consequence of the above equations, we have

$$\begin{aligned} f(w,\eta) = f_{00}(y) + f'_{00}(y)\frac{\tau(\zeta+\bar{\zeta})}{2i} + f_{10}(y)\zeta - if_{10}(y)\tau \\ + f_{01}(y)\bar{\zeta} + if_{01}(y)\tau + f_{11}(y)\bar{\zeta}\zeta + if_{11}(y)\tau(\zeta+\bar{\zeta}). \end{aligned}$$

Therefore, a function is invariant under the action of $\mathbb{R}^{1|1}$, if this function satisfies equation (4.2) and both

$$f_{10}(y) = f_{01}(y) \quad \text{and} \quad f_{11}(y) = \frac{f'_{00}(y)}{2}. \qquad \square$$

5. Toeplitz operator with symbols invariants under the action of $\mathbb{R}^{1|1}$

We take a bounded super function F and we assume that F is invariant under the action of $\mathbb{R}^{1|1}$, then we have that the Toeplitz operator with symbol F has the follow form

$$\begin{aligned} T_F^{(\nu)}(\Psi)(w,\eta) = \frac{\nu-1}{\pi}\int\limits_{\mathbb{H}} f_0(\mathrm{Im}(z))\psi_0(z)(2\mathrm{Im}(z))^{\nu-2}\left(\frac{w-\bar{z}}{i}\right)^{-\nu} dz \\ + \frac{1}{\pi}\int\limits_{\mathbb{H}} \frac{f_0'(\mathrm{Im}(z))}{2}\psi_0(z)(2\mathrm{Im}(z))^{\nu-1}\left(\frac{w-\bar{z}}{i}\right)^{-\nu} dz \\ + \frac{1}{\pi}\int\limits_{\mathbb{H}} f_1(\mathrm{Im}(z))\psi_1(z)(2\mathrm{Im}(z))^{\nu-1}\left(\frac{w-\bar{z}}{i}\right)^{-\nu} dz \\ + \left(\frac{\nu}{\pi}\int\limits_{\mathbb{H}} f_1(\mathrm{Im}(z))\psi_0(z)(2\mathrm{Im}(z))^{\nu-1}\left(\frac{w-\bar{z}}{i}\right)^{-(\nu+1)} dz\right)\eta \\ + \left(\frac{\nu}{\pi}\int\limits_{\mathbb{H}} f_0(\mathrm{Im}(z))\psi_1(z)(2\mathrm{Im}(z))^{\nu-1}\left(\frac{w-\bar{z}}{i}\right)^{-(\nu+1)} dz\right)\eta. \end{aligned} \tag{5.1}$$

In [4] it was shown that if $\psi \in H^2_\nu(\mathbb{H})$ then it has a representation in the form of a Fourier integral

$$\psi(x+iy) = \frac{1}{\sqrt{\Gamma(\nu)}}\int_{\mathbb{R}_+} t^{\frac{\nu-1}{2}}\phi(t)e^{it(x+iy)}dt$$

where $\phi \in L_2(\mathbb{R}_+)$.

We know that if $\psi_0 \in H^2_\nu(\mathbb{H})$ and $\psi_1 \in H^2_{\nu+1}(\mathbb{H})$ then by the above equation we have

$$\psi_0(x+iy) = \frac{1}{\sqrt{\Gamma(\nu)}} \int_{\mathbb{R}_+} t^{\frac{\nu-1}{2}} \phi_0(t) e^{it(x+iy)} dt \tag{5.2}$$

$$\psi_1(x+iy) = \frac{1}{\sqrt{\Gamma(\nu+1)}} \int_{\mathbb{R}_+} t^{\frac{\nu}{2}} \phi_1(t) e^{it(x+iy)} dt \tag{5.3}$$

where $\phi_0, \phi_1 \in L_2(\mathbb{R}_+)$.

Substituting the above equations in (5.1) we obtain

$$\begin{aligned} T_F^{(\nu)}(\Psi)(w,\eta) = & \left(\frac{\nu-1}{\pi} \int\limits_{\mathbb{R}_+} \left(f_0(y) + \frac{y f_0'(y)}{\nu-1} \right) \frac{1}{\sqrt{\Gamma(\nu)}} \int_{\mathbb{R}_+} t^{\frac{\nu-1}{2}} \phi_0(t) e^{-ty} (2y)^{\nu-2} \right. \\ & \left. \times \int\limits_{\mathbb{R}} \left(\frac{w-(x-iy)}{i} \right)^{-\nu} e^{itx} dx dt dy \right) \\ & + \left(\frac{1}{\pi} \int\limits_{\mathbb{R}_+} f_1(y) \frac{1}{\sqrt{\Gamma(\nu+1)}} \int_{\mathbb{R}_+} t^{\frac{\nu}{2}} \phi_1(t) e^{-ty} (2y)^{\nu-1} \right. \\ & \left. \times \int\limits_{\mathbb{R}} \left(\frac{w-(x-iy)}{i} \right)^{-\nu} e^{itx} dx dt dy \right) \\ & + \left(\frac{\nu}{\pi} \int\limits_{\mathbb{R}_+} f_1(y) \frac{1}{\sqrt{\Gamma(\nu)}} \int_{\mathbb{R}_+} t^{\frac{\nu-1}{2}} \phi_0(t) e^{-ty} (2y)^{\nu-1} \right. \\ & \left. \times \int\limits_{\mathbb{R}} \left(\frac{w-(x-iy)}{i} \right)^{-(\nu+1)} e^{itx} dx dt dy \right) \eta \\ & + \left(\frac{\nu}{\pi} \int\limits_{\mathbb{R}_+} f_0(y) \frac{1}{\sqrt{\Gamma(\nu+1)}} \int_{\mathbb{R}_+} t^{\frac{\nu}{2}} \phi_1(t) e^{-ty} (2y)^{\nu-1} \right. \\ & \left. \times \int\limits_{\mathbb{R}} \left(\frac{w-(x-iy)}{i} \right)^{-(\nu+1)} e^{itx} dx dt dy \right) \eta. \end{aligned}$$

Using the formula, see [7] 3.382.6,

$$\int\limits_{\mathbb{R}} (i\beta - x)^{-\nu} e^{itx} dx = \frac{2\pi\beta t^{\nu-1} e^{-\beta t}}{i^\nu \Gamma(\nu)} \text{ where } t > 0$$

we have that $T_F^{(\nu)}(\Psi)(w,\eta)$ is equal to

$$
\begin{aligned}
&= \frac{\nu-1}{\pi}\int_{\mathbb{R}_+}\left(f_0(y)+\frac{yf_0'(y)}{\nu-1}\right)\frac{1}{\sqrt{\Gamma(\nu)}}\int_{\mathbb{R}_+} t^{\frac{\nu-1}{2}}\phi_0(t)e^{-ty}(2y)^{\nu-2} \\
&\quad\times\left(\frac{1}{\Gamma(\nu)}2\pi t^{\nu-1}e^{itw}e^{-ty}\right)dtdy \\
&\quad+\frac{1}{\pi}\int_{\mathbb{R}_+} f_1(y)\frac{1}{\sqrt{\Gamma(\nu+1)}}\int_{\mathbb{R}_+} t^{\frac{\nu}{2}}\phi_1(t)e^{-ty}(2y)^{\nu-1}\left(\frac{1}{\Gamma(\nu)}2\pi t^{\nu-1}e^{itw}e^{-ty}\right)dtdy \\
&\quad+\frac{\nu}{\pi}\int_{\mathbb{R}_+} f_1(y)\frac{1}{\sqrt{\Gamma(\nu)}}\int_{\mathbb{R}_+} t^{\frac{\nu-1}{2}}\phi_0(t)e^{-ty}(2y)^{\nu-1}\left(\frac{1}{\Gamma(\nu+1)}2\pi t^{\nu}e^{itw}e^{-ty}\right)dtdy\eta \\
&\quad+\frac{\nu}{\pi}\int_{\mathbb{R}_+} f_0(y)\frac{1}{\sqrt{\Gamma(\nu+1)}}\int_{\mathbb{R}_+} t^{\frac{\nu}{2}}\phi_1(t)e^{-ty}(2y)^{\nu-1}\left(\frac{1}{\Gamma(\nu+1)}2\pi t^{\nu}e^{itw}e^{-ty}\right)dtdy\eta.
\end{aligned}\tag{5.4}
$$

Equivalently,

$$
\begin{aligned}
&= \frac{1}{\sqrt{\Gamma(\nu)}}\int_{\mathbb{R}_+} t^{\frac{\nu-1}{2}}\phi_0(t)\left(\frac{t^{\nu-1}}{\Gamma(\nu-1)}\right. \\
&\qquad\left.\times\int_{\mathbb{R}_+}\left(f_0(y)+\frac{yf_0'(y)}{\nu-1}\right)e^{-2ty}(2y)^{\nu-2}2dy\right)e^{itw}dt \\
&\quad+\frac{1}{\sqrt{\Gamma(\nu)}}\int_{\mathbb{R}_+} t^{\frac{\nu-1}{2}}\phi_1(t)\left(\frac{1}{\nu^{\frac{1}{2}}}\frac{t^{\nu-\frac{1}{2}}}{\Gamma(\nu)}\times\int_{\mathbb{R}_+} f_1(y)e^{-2ty}(2y)^{\nu-1}2dy\right)e^{itw}dt \\
&\quad+\frac{1}{\sqrt{\Gamma(\nu+1)}}\int_{\mathbb{R}_+} t^{\frac{\nu}{2}}\phi_0(t)\left(\nu^{\frac{1}{2}}\frac{t^{\nu-\frac{1}{2}}}{\Gamma(\nu)}\int_{\mathbb{R}_+} f_1(y)e^{-2ty}(2y)^{\nu-1}2dy\right)e^{itw}dt\eta \\
&\quad+\frac{1}{\sqrt{\Gamma(\nu+1)}}\int_{\mathbb{R}_+} t^{\frac{\nu}{2}}\phi_1(t)\left(\frac{t^{\nu}}{\Gamma(\nu)}\int_{\mathbb{R}_+} f_0(y)e^{-2ty}(2y)^{\nu-1}2dy\right)e^{itw}dt\eta.
\end{aligned}\tag{5.5}
$$

Using the property of Laplace transform with respect to derivative

$$\int_{\mathbb{R}_+} f'(y)e^{-ty}dy = t\int_{\mathbb{R}_+} f(y)e^{-ty}dy - f(0)$$

we obtain

$$\frac{t}{\nu-1}\int_{\mathbb{R}_+} f_0\left(\frac{y}{2}\right)y^{\nu-1}e^{-ty}dy = \int_{\mathbb{R}_+}\left(f_0\left(\frac{y}{2}\right)+\frac{yf_0'(\frac{y}{2})}{2(\nu-1)}\right)y^{\nu-2}e^{-ty}dy. \tag{5.6}$$

Substituting (5.6) in (5.5), we have that if $\Psi = \psi_0(w) + \psi_1(w)\eta$ where ψ_i are given by (5.2) and (5.3), and F is $\mathbb{R}^{1|1}$-invariant, then the Toeplitz operator with symbol F over Ψ is given by

$$T_F^\nu(\Psi)(w,\eta) = \frac{1}{\sqrt{\Gamma(\nu)}}\int_{\mathbb{R}_+} t^{\frac{\nu-1}{2}}\left(\phi_0(t)\gamma_{[f_0,\nu]}(t) + \frac{\phi_1(t)\gamma_{[f_1,\nu]}(t)}{t^{\frac{1}{2}}\nu^{\frac{1}{2}}}\right)e^{itw}dt \quad (5.7)$$
$$+ \frac{\eta}{\sqrt{\Gamma(\nu+1)}}\int_{\mathbb{R}_+} t^{\frac{\nu}{2}}\left(\frac{\nu^{\frac{1}{2}}\phi_0(t)\gamma_{[f_1,\nu]}(t)}{t^{\frac{1}{2}}} + \phi_1(t)\gamma_{[f_0,(\nu)]}(t)\right)e^{itw}dt$$

where

$$\gamma_{[f,\nu]}(t) = \frac{t^\nu}{\Gamma(\nu)}\int\limits_{\mathbb{R}_+} f(\frac{y}{2})e^{-ty}y^{\nu-1}dy. \text{ where } t>0.$$

It is clear that if f is bounded then $\gamma_{[f,\nu]}(t)$ is also bounded over $\mathbb{R}_+$. Therefore, $\gamma_{[f_i,\nu]}\phi_j \in L_2(\mathbb{R}_+)$ for $i,j = 0,1$.

Theorem 5.1. *The Toeplitz algebra generated by all super Toeplitz operators whose symbols are invariant under the action of* $\mathbb{R}^{1|1}$ *is commutative.*

Proof. Consider the elements in $H^2_\nu(\mathbb{H}^{1|1})$ of the form

$$\Psi_+(w,\eta) = \int_{\mathbb{R}_+}\left(\frac{t^{\frac{\nu-1}{2}}}{\sqrt{\Gamma(\nu)}} + \nu^{\frac{1}{2}}\frac{t^{\frac{\nu}{2}}\eta}{\sqrt{\Gamma(\nu+1)}}\right)\phi(t)e^{itw}dt$$
$$\Psi_-(w,\eta) = \int_{\mathbb{R}_+}\left(\frac{t^{\frac{\nu-1}{2}}}{\sqrt{\Gamma(\nu)}} - \nu^{\frac{1}{2}}\frac{t^{\frac{\nu}{2}}\eta}{\sqrt{\Gamma(\nu+1)}}\right)\phi(t)e^{itw}dt$$

where $\phi \in L_2(\mathbb{R})$. It is clear that $\{\Psi_+, \Psi_-\}$ is a base of $H^2_\nu(\mathbb{H}^{1|1})$.

Let F be a $\mathbb{R}^{1|1}$-invariant super function, then using the form of the Toeplitz operator with symbol F given by (5.7) over Ψ_+ and Ψ_- we obtain

$$T_F^\nu(\Psi_+)(w,\eta) = \int_{\mathbb{R}_+}\left(\frac{t^{\frac{\nu-1}{2}}}{\sqrt{\Gamma(\nu)}} + \nu^{\frac{1}{2}}\frac{t^{\frac{\nu}{2}}\eta}{\sqrt{\Gamma(\nu+1)}}\right)[\gamma_{[f_0,\nu]}(t) + t^{-\frac{1}{2}}\gamma_{[f_1,\nu]}(t)]\phi_0(t)e^{itw}dt$$
$$T_F^\nu(\Psi_-)(w,\eta) = \int_{\mathbb{R}_+}\left(\frac{t^{\frac{\nu-1}{2}}}{\sqrt{\Gamma(\nu)}} - \nu^{\frac{1}{2}}\frac{t^{\frac{\nu}{2}}\eta}{\sqrt{\Gamma(\nu+1)}}\right)[\gamma_{[f_0,\nu]}(t) - t^{-\frac{1}{2}}\gamma_{[f_1,\nu]}(t)]\phi_0(t)e^{itw}dt$$

where $t > 0$.

If F, G are $\mathbb{R}^{1|1}$-invariant super functions, then using the above formulas we have that the composition of the Toeplitz operators with symbols F and G is given by

$$T_G^\nu(T_F^\nu(\Psi_+))(w,\eta) = \int_{\mathbb{R}_+} \left(\frac{t^{\frac{\nu-1}{2}}}{\sqrt{\Gamma(\nu)}} + \nu^{\frac{1}{2}} \frac{t^{\frac{\nu}{2}}\eta}{\sqrt{\Gamma(\nu+1)}} \right) [\gamma_{[f_0,\nu]}(t) + t^{-\frac{1}{2}}\gamma_{[f_1,\nu]}(t)][\gamma_{[g_0,\nu]}(t) + t^{-\frac{1}{2}}\gamma_{[g_1,\nu]}(t)]\phi_0(t)e^{itw}dt$$

$$T_G^\nu(T_F^\nu(\Psi_-))(w,\eta) = \int_{\mathbb{R}_+} \left(\frac{t^{\frac{\nu-1}{2}}}{\sqrt{\Gamma(\nu)}} - \nu^{\frac{1}{2}} \frac{t^{\frac{\nu}{2}}\eta}{\sqrt{\Gamma(\nu+1)}} \right) [\gamma_{[f_0,\nu]}(t) - t^{-\frac{1}{2}}\gamma_{[f_1,\nu]}(t)][\gamma_{[g_0,\nu]}(t) - t^{-\frac{1}{2}}\gamma_{[g_1,\nu]}(t)]\phi_0(t)e^{itw}dt,$$

where $t > 0$. Therefore, by the above equations we obtain the desired result. □

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Measure Characterization involving the Limiting Eigenvalue Distribution for Schrödinger Operators on S^2

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Abstract. Knowing a result called the Limiting Eigenvalue Distribution (LED) on S^2 it is posible to stablish a natural way to define a Baire measure related to the Radon Transform of a potential V on the sphere S^2.

The aim of this work is to give some results and examples of how we can characterize some properties of the potential V in order to determine what kind of Baire Measure we can expect.

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1. Introduction

Let $H = -\Delta + V$ be a Schrödinger operator in $L^2(S^n)$, where S^n is the n unit-sphere in $\mathbb{R}^n$. The potential V is a real continuous function and $-\Delta$ is the Laplace-Beltrami Operator.

It is well known that in the case $V = 0$, H has eigenvalues:

$$\lambda_\ell = \ell(\ell + 1),$$

with each λ_ℓ having increasing degeneracy and $d_\ell = O(\ell^{n-1})$.

If $V \neq 0$ the eigenvalue λ_ℓ splits into a cluster of eigenvalues contained in an interval of radius $\|V\|_\infty$, with center in λ_ℓ.

Another way of expressing this is in terms of the "spectral shifts", $\mu_{\ell,\nu}$, i.e., the distance between the center λ_ℓ and the split eigenvalue $\lambda_{\ell,\nu}$, so that we can write

$$\lambda_{\ell,\nu} = \lambda_\ell + \mu_{\ell,\nu} \quad -\ell \leq \nu \leq \ell.$$

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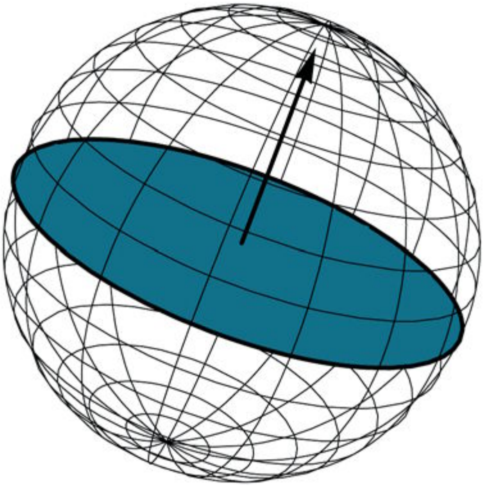

FIGURE 1. A geodesic on the sphere with its normal vector.

An interesting problem is the study of the asymptotic distribution of the cluster of eigenvalues (or the spectral shifts).

Several authors gave the answer, which is known as the Limiting Eigenvalue Distribution result (LED). Refering to the point of view on the asymptotic distribution of the eigenvalues for certain homogeneus spaces the reader can check [9], [3], [10]. The answer from the point of view on the asymptotics of the spectral shifts can be found in [1], [4].

We proceed to describe the LED result following the work of Guillemin-Sternberg [3] but particularizing it on the S^2 sphere.

Theorem 1.1 (Limiting Eigenvalue Distribution Theorem on S^2). *Let $\mathcal{S}$ be the Schwartz space on the real line. Let $\Psi \in \mathcal{S}$. Then*

$$\lim_{\ell\to\infty} \frac{1}{d_\ell} \sum_{\nu} \Psi(\lambda_{\ell,\nu} - \ell(\ell+1)) = \frac{1}{4\pi} \int_{S^2} \Psi(\hat{V}(\gamma))\, \mathrm{d}S\,, \tag{1.1}$$

where $\hat{V}(\gamma)$ is the Radon transform of the potential V in the space of geodesics in the sphere, G_2. That is to say, for $\gamma \in G_2$,

$$\hat{V}(\gamma) := \frac{1}{2\pi} \int V(\gamma(s))\, \mathrm{d}s. \tag{1.2}$$

We can identify G_2 with S^2 if we identify antipodal points in S^2. In other words, if $\omega \in S^2/\mathbb{Z}_2$ and considering γ the geodesic in the plane orthogonal of ω, (see Figure 1), then

$$\hat{V}(\omega) := \frac{1}{2\pi} \int_{\gamma\cdot\omega=0} V(\gamma(s))\, \mathrm{d}s. \tag{1.3}$$

Applying the Riesz-Markov representation theorem we can also write the result of limiting eigenvalue distribution theorem as follows:

$$\lim_{\ell\to\infty}\frac{1}{d_\ell}\sum_{\nu}\Psi(\lambda_{\ell,\nu}-\ell(\ell+1))=\int\Psi(\lambda)\,\mathrm{d}\mu_V(\lambda) \tag{1.4}$$

where μ_V is a Baire Measure, defined for the eigenvalue clusters.

We would like to give certain "characterizations" of the potential V in order to have several types of measures μ_V, such as discrete, absolutely or singular continuous. In the following discusion we give some answers to this respect.

2. Results

Our first characterization concerns the odd part V_{odd} of the potential V.

Proposition 2.1. *Consider V a continuous funtion on the sphere S^2. Then the Baire Measure corresponding to V_{odd}, $\mu_{V_{\text{odd}}}$, is a Dirac Measure.*

Proof. Decompose the potential in the form $V=V_{\text{odd}}+V_{\text{even}}$, where V_{odd}, V_{even} are the odd and even part of V. We can easily find that $\mu_{V_{\text{odd}}}$ is pure point, as a consecuence of the definition of the Radon Transform on S^2. □

With this result we can center the discussion exclusively on even potentials V.

Proposition 2.2. *Consider V a continuous funtion on the sphere S^2. If we supose that V is constant in a band B on S^2 of the form $S^2\sim\{z\in S^2\mid 0<\phi\le 2\pi,\ -\theta_0<\theta<\theta_0\}$ (ϕ is the azimutal angle, θ the inclination angle, and θ_0 is a given inclination angle in $(0,\frac{\pi}{2})$), then μ_V has non trivial discrete part (pure point) and the corresponding cumulative distribution function is not continuous.*

Proof. If $V=b\in B$ we have by definition $\hat{V}=b\in B$, but then $|\hat{V}^{-1}(\{b\})|=|B|$, where $|\cdot|$ is the Lebesgue measure on S^2 and by hypotesis $|B|\neq 0$.

Now, since μ_V is a Baire Measure we can associate a cumulative distribution function $F(x)$ in the standard way:

$$F(x)=\mu_V(-\infty,x) \tag{2.1}$$

But then, since

$$F(b)-F(b_-)=\lim_{n\to\infty}\mu_V(b-\frac{1}{n},b)=\mu_V(\{b\})\neq 0\,, \tag{2.2}$$

we conclude that $F(x)$ is not continuous.

□

In order to give a complete characterization of smooth potentials we have to stablish the following important result, concerning to its Radon transform.

Theorem 2.3. *If V is differentiable on S^2 then $\hat{V}$ is differentiable in $S^2/\mathbb{Z}_2$.*

Proof. We are going to give a proof without coordinates.

First of all, define the **unit tangent space** of S^2

$$(TS^2)^1 = \{v_p \in T_pS^2| \; \|v_p\| = 1\,,\; p \in S^2\}\,. \tag{2.3}$$

Next, define the differentiable mapping $\varphi : (TS^2)^1 \to S^2$ by $\varphi(v_p) = v$.

Also, given the differentiable function $V : S^2 \to \mathbb{R}$, consider the pullback $\varphi^*V : (TS^2)^1 \to \mathbb{R}$ defined by $\varphi^*V = V \circ \varphi$.

On the other hand, consider π, the canonical projection $\pi : (TS^2)^1 \longrightarrow S^2$, which is a fiber bundle with total space $(TS^2)^1$, base S^2, and each fiber isomorphic to S^1.

Since $(TS^2)^1$ is locally a cartesian product then, again locally, $\varphi^*V = f(p, \theta)$ with $p \in S^2$ and $\theta \in S^1$.

Since each φ can be considered a geodesic on S^2, we can check that $\langle \varphi^*V \rangle = \langle V \rangle$, where on the l.h.s. $\langle\;\rangle$ denotes average over the fiber S^1 and on the r.h.s. it denotes average over geodesics.

Since φ^*V is differentiable then so is

$$\langle \varphi^*V \rangle(p) = \frac{1}{2\pi} \int_0^{2\pi} f(p, \theta)\, d\theta,$$

and then so is $\langle V \rangle$.

Hence, since $\langle V \rangle$ is $\mathbb{Z}_2$-invariant, then it drops to $S^2/\mathbb{Z}_2$ and therefore we can identify $\hat{V} = \langle V \rangle$. □

Once we have established this theorem it is posible to give a more general characterization as a consecuence of the Radon-Nikodym Theorem [7] and the previuous results.

Theorem 2.4. *Supose that V is differentiable. Then the associated Baire Measure μ_V has an absolutely continuous part plus a pure point part corresponding to regions E of positive measure where $\hat{V} = C$, C a constant.*

In the case of a μ_V singular continuous we have the following interesting example.

2.1. Singular continuous measure

In what follows we are going to consider the case when V is zonal (axially symmetric, with respect to the z axis). Then it is easy to prove the following result.

Proposition 2.5. *If V is a continuous function on S^2 and zonal, then $\hat{V}$ is also zonal.*

We are going to note that, if we charaterize the associate cumulative distribution function $F(x)$ in order to make μ_V singular continuous nontrivial, then we can observe certain geometric behavior on the asymptotic distribution of the spectral shifts of the potential V.

First of all we are going to consider the following generalized Cantor set, for middle-odd parts: [1]
Let $I := [0, 1]$ the unit interval. Let $k = 2n + 1,\ n = 1, 2, 3 \ldots.$
Define $k_1 := \frac{[\frac{k}{2}]}{k}$, $k_2 := \frac{[\frac{k}{2}]+1}{k}$, where $[\frac{k}{2}]$ is the integer part of $\frac{k}{2}$. Then, the construction of the generalized cantor set will be the standar, i.e.:

- In the first step take,
$$C_1 := I - (k_1, k_2),$$
- in the second step,
$$C_2 := C_1 - (k_1^2, k_1 k_2) \cup (k_1^2 + k_2, k_1 k_2 + k_2)$$
- in the third,
$$\begin{aligned} C_3 := C_2 - &(k_1^3, k_1^2 k_2) \cup (k_1^3 + k_1 k_2, k_1^2 k_2 + k_1 k_2) \\ &\cup (k_1^3 + k_2, k_1^2 k_2 + k_2) \cup (k_1^3 + k_1 k_2 + k_2, k_1^2 k_2 + k_1 k_2 + k_2), \end{aligned} \tag{2.4}$$
- and so on ...

The generalized Cantor set is then defined by $C := \cup_n C_n$. As in the case of the middle-third Cantor set, it is posible to prove that the generalized Cantor set is a noncountable, measure cero (with respecto to the Lebesgue measure λ in $\mathbb{R}$) set in $\mathbb{R}$.

Now consider the complement of C, $I - C$ which can be seen as:

$$\begin{aligned} I - C &= (k_1, k_2) \cup (k_1^2, k_1 k_2) \cup (k_1^2 + k_2, k_1 k_2 + k_2) \cup (k_1^3, k_1^2 k_2) \cup \cdots \\ &= \left(\frac{[\frac{k}{2}]}{k}, \frac{[\frac{k}{2}] + 1}{k} \right) \cup \left(\frac{[\frac{k}{2}]^2}{k^2}, \frac{([\frac{k}{2}] + 1)[\frac{k}{2}]}{k^2} \right) \\ &\quad \cup \left(\frac{[\frac{k}{2}]^2}{k^2} + \frac{[\frac{k}{2}] + 1}{k}, \frac{([\frac{k}{2}] + 1)[\frac{k}{2}]}{k^2} + \frac{[\frac{k}{2}] + 1}{k} \right) \cup \left(\frac{[\frac{k}{2}]^3}{k^3}, \frac{[\frac{k}{2}] + 1}{k} \frac{[\frac{k}{2}]^2}{k^2} \right) \cup \cdots \end{aligned}$$

Then define the cumulative distribution function, $F(x)$, as follows:

$$\begin{aligned} F(x) &= \frac{1}{2}, \quad x \in \left(\frac{[\frac{k}{2}]}{k}, \frac{[\frac{k}{2}] + 1}{k} \right), \\ F(x) &= \frac{1}{2^2}, \quad x \in \left(\frac{[\frac{k}{2}]^2}{k^2}, \frac{([\frac{k}{2}] + 1)[\frac{k}{2}]}{k^2} \right), \\ F(x) &= \frac{3}{2^2}, \quad x \in \left(\frac{[\frac{k}{2}]^2}{k^2} + \frac{[\frac{k}{2}] + 1}{k}, \frac{([\frac{k}{2}] + 1)[\frac{k}{2}]}{k^2} + \frac{[\frac{k}{2}] + 1}{k} \right), \\ F(x) &= \frac{1}{2^3}, \quad x \in \left(\frac{[\frac{k}{2}]^3}{k^3}, \frac{[\frac{k}{2}] + 1}{k} \frac{[\frac{k}{2}]^2}{k^2} \right), \end{aligned}$$

and so on ...

[1] We can also construct the generalized cantor set for middle-even parts but we lose symmetry.

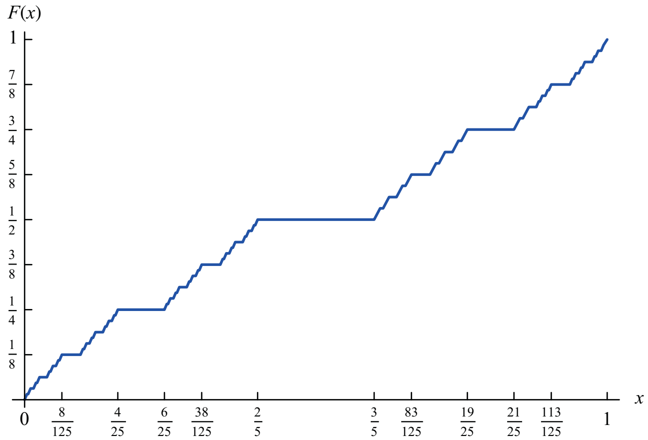

FIGURE 2. The Cantor Function for a middle-fifth Cantor set.

And finally consider the continuous extension of $F(x)$. (See Figure 2 for the case $k = 5$.) Then, $F(x)$ is a nonconstant, continuous function. By construction the corresponding measure μ is concentrated in the Cantor set C. Then, μ is singular continuous with respect to the Lebesgue measure on the real line λ.

Consider, as we are interested in, that $\mu = \mu_V$. The property that μ_V is concentrated on C is now very relevant, because this shows that the Radon transform of V is supported on the Cantor set C. But we know that the image of the Radon transform looks like the spectral shifts $\mu_{\ell,\nu}$ (unless it is $O(\ell^{-2})$), as we can check in [4] (Theorem 1) for continuous potentials.

We can conclude that in the limit when $\ell \to \infty$ the spectral shifts (in the limiting cluster) are distributed in a very similar way as the Cantor set C. In this manner we already have a way of checking how the nature of the measure μ_V determines a geometric property on the distribution of the limiting spectral cluster of V.

3. Future work

We think is important to have a more general result for potentials V, continuous but not necessarily smooth (for example, Hölder Continuous of order α, $0 < \alpha \leq 1$).

Also, we are really interested in results similar to the case of singular continuous μ_V but considering non-zonal potentials.

Finally, we are looking for examples of measures μ_V that exhibit mixtures between continuous parts and what does this mean about the nature of V.

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Relations Among Various Versions of the Segal-Bargmann Transform

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Abstract. We present various relations among Versions A, B and C of the Segal-Bargmann transform. We get results for the Segal-Bargmann transform associated to a Coxeter group acting on a finite-dimensional Euclidean space. Then analogous results are shown for the Segal-Bargmann transform of a connected, compact Lie group for all except one of the identities established in the Coxeter case. A counterexample is given to show that the remaining identity from the Coxeter case does not have an analogous identity for the Lie group case. A major result is that in both contexts the Segal-Bargmann transform for Version C is determined by that for Version A.

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1. A Brief Introduction

We recall quickly some notations and definitions from [20]. Many definitions and details are not presented here. We also advise the reader that our normalizations are not standard.

A root system is a certain finite subset $\mathcal{R}$ of nonzero vectors of $\mathbb{R}^N$ where $N \geq 1$ is an integer. It turns out that the finite set of reflections associated to these vectors (orthogonal reflection in the hyperplane perpendicular to each vector) generates a finite subgroup, known as the Coxeter group, of the orthogonal group of $\mathbb{R}^N$. A multiplicity function is a function $\mu : \mathcal{R} \to \mathbb{C}$ invariant under the action of the Coxeter group. We always will assume that the multiplicity function satisfies $\mu \geq 0$. This condition is sufficient for the existence of the Segal-Bargmann spaces considered and for the various properties that we shall use.

We will take $t > 0$ (Planck's constant) fixed throughout this paper.

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We will use the holomorphic Dunkl kernel function $E_\mu : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$, which for all $z, w \in \mathbb{C}^N$ satisfies $E_\mu(z,w) = E_\mu(w,z)$, $E_\mu(z,0) = 1$ and $E_\mu(z^*,z) \geq 0$ among many other properties. When $\mu \equiv 0$, we have $E_\mu(z,w) = e^{z\cdot w}$.

We will also be using the analytic continuation of the Dunkl heat kernel, which is given for $z, w \in \mathbb{C}^N$ by

$$\rho_{\mu,t}(z,w) = e^{-(z^2+w^2)/2t} E_\mu\left(\frac{z}{t^{1/2}}, \frac{w}{t^{1/2}}\right). \tag{1.1}$$

This kernel arises in the solution of the initial value problem of the heat equation associated with the Dunkl Laplacian operator. (See [13].)

We next define the kernel functions of the versions of the Segal-Bargmann transform associated to a Coxeter group for $z \in \mathbb{C}^N$ and $q \in \mathbb{R}^N$ by

$$A_{\mu,t}(z,q) := e^{-z^2/2t - q^2/4t} E_\mu\left(\frac{z}{t^{1/2}}, \frac{q}{t^{1/2}}\right) \tag{1.2}$$

and

$$B_{\mu,t}(z,q) := \frac{\rho_{\mu,t}(z,q)}{\rho_{\mu,t}(0,q)} \tag{1.3}$$

and

$$C_{\mu,t}(z,q) := \rho_{\mu,t}(z,q). \tag{1.4}$$

See [1], [4] and [15] for the origins of this theory in the case $\mu \equiv 0$.

The versions of the Segal-Bargmann transform are given as follows. (See [2], [5], [16], [17] and [20].) Versions A and C are defined by

$$A_{\mu,t} f(z) := \int_{\mathbb{R}^N} \mathrm{d}\omega_{\mu,t}(q)\, A_{\mu,t}(z,q) f(q)$$

and

$$C_{\mu,t} f(z) := \int_{\mathbb{R}^N} \mathrm{d}\omega_{\mu,t}(q)\, C_{\mu,t}(z,q) f(q)$$

respectively, where $z \in \mathbb{C}^N$, $f \in L^2(\mathbb{R}^N, \omega_{\mu,t})$ and $\omega_{\mu,t}$ is the density of a measure on $\mathbb{R}^N$. Version B is defined by

$$B_{\mu,t} f(z) := \int_{\mathbb{R}^N} \mathrm{d}m_{\mu,t}(q)\, B_{\mu,t}(z,q) f(q),$$

where $z \in \mathbb{C}^N$, $f \in L^2(\mathbb{R}^N, m_{\mu,t})$ and $m_{\mu,t}$ is the density of a measure on $\mathbb{R}^N$.

Associated to these versions there are reproducing kernel Hilbert spaces of holomorphic functions $f : \mathbb{C}^N \to \mathbb{C}$, denoted $\mathcal{A}_{\mu,t}$, $\mathcal{B}_{\mu,t}$ and $\mathcal{C}_{\mu,t}$ respectively, such that

$$\begin{aligned} A_{\mu,t} &: L^2(\mathbb{R}^N, \omega_{\mu,t}) \to \mathcal{A}_{\mu,t} \\ B_{\mu,t} &: L^2(\mathbb{R}^N, m_{\mu,t}) \to \mathcal{B}_{\mu,t} \\ C_{\mu,t} &: L^2(\mathbb{R}^N, \omega_{\mu,t}) \to \mathcal{C}_{\mu,t} \end{aligned}$$

are unitary isomorphisms. It turns out that $\mathcal{A}_{\mu,t} = \mathcal{B}_{\mu,t}$ as Hilbert spaces.

The holomorphic function $\rho_{\mu,t} : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ in our opinion is not a fundamental object. Rather we view $\sigma_{\mu,t}(q) := \rho_{\mu,t}(0,q) = e^{-q^2/2t}$ for $q \in \mathbb{R}^N$ as the fundamental Dunkl heat kernel, even though it does not depend on μ. Then this one-variable kernel $\sigma_{\mu,t} : \mathbb{R}^N \to (0, \infty)$ gives rise to the two-variable kernel

$$\rho_{\mu,t} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \tag{1.5}$$

using a generalized (or Dunkl) translation operator, denoted $\mathcal{T}_{\mu,x}$, via the equation

$$\rho_{\mu,t}(x, q) = \mathcal{T}_{\mu,x}\sigma_{\mu,t}(q)$$

for all $x, q \in \mathbb{R}^N$. (See [20] for more details, including definitions and proofs.) In our conventions, note that for $\mu \equiv 0$ we have $\rho_{0,t}(x, q) = \sigma_{0,t}(q - x)$. Finally the function $\rho_{\mu,t} : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ is obtained from (1.5) by analytic continuation.

For more details about this background material see references [2], [3], [13], [14] and [20], while for other related research in Segal-Bargmann analysis see [7], [8], [11], [12], [19] and [21].

2. Coxeter group case

We proved the following relation between the kernel functions for the A Version and the C Version of the Segal-Bargmann transform associated to a Coxeter group in [20], namely,

$$C_{\mu,t}(z, q) = A_{\mu,t}(0, q)A_{\mu,t}(z, q) \tag{2.1}$$

for $z \in \mathbb{C}^N$ and $q \in \mathbb{R}^N$. The reader can readily verify this using the definitions in the previous section. As an immediate consequence we have this identity:

$$\begin{aligned} C_{\mu,t}\psi(z) &= \int_{\mathbb{R}^N} \mathrm{d}\omega_{\mu,t}(q)\, C_{\mu,t}(z, q)\psi(q) \\ &= \int_{\mathbb{R}^N} \mathrm{d}\omega_{\mu,t}(q)\, A_{\mu,t}(z, q)A_{\mu,t}(0, q)\psi(q) \end{aligned} \tag{2.2}$$

for all $\psi \in L^2(\mathbb{R}^N, \omega_{\mu,t})$ and all $z \in \mathbb{C}^N$.

So, we have represented the unitary operator $C_{\mu,t}$ as the composition of two operators: the first is the operator (denoted by M_t) of multiplication by the bounded function $A_{\mu,t}(0, q) = e^{-q^2/4t}$, and the second is the unitary operator $A_{\mu,t}$. In other words we can write (2.2) as

$$C_{\mu,t} = A_{\mu,t}M_t. \tag{2.3}$$

As far as we are aware this representation is new, even in the case when $\mu \equiv 0$. The boundedness of the function $A_{\mu,t}(0, q)$, where $q \in \mathbb{R}^N$, is essential since this gives us that M_t is an operator from $L^2(\mathbb{R}^N, \omega_{\mu,t})$ to *itself*. Therefore, the second operator $A_{\mu,t}$ in (2.3) is acting on the space where it is a unitary operator. Moreover, the operator norm of M_t satisfies $||M_t|| = \sup_{q\in\mathbb{R}^N}(e^{-q^2/4t}) = 1$.

Though this representation of $C_{\mu,t}$ is similar to a Toeplitz operator, it is decidedly different. Here we have multiplication by a bounded function followed by a specific unitary operator, while a Toeplitz operator is multiplication by a bounded function followed by a specific projection operator.

In the case $\mu \equiv 0$, it is known that $\mathcal{C}_{\mu,t} \subset \mathcal{A}_{\mu,t}$, a bounded inclusion. When $\mu \equiv 0$ these two reproducing kernel Hilbert spaces can alternatively be defined in terms of measures on $\mathbb{C}^N$. Then the bounded inclusion follows for example from the formulas for these measures. (See [6], p. 51, where the formulas given there for these measures for $N = 1$ also hold for $N > 1$.) The generalization of this to the present context is the next result.

Theorem 2.1. *We have the contractive* (*in particular, bounded*) *inclusion*

$$\mathcal{C}_{\mu,t} \subset \mathcal{A}_{\mu,t}.$$

Proof. Using (2.3), we have that

$$\mathcal{C}_{\mu,t} = \mathrm{Ran}(C_{\mu,t}) = \mathrm{Ran}(A_{\mu,t} M_t) \subset \mathrm{Ran}(A_{\mu,t}) = \mathcal{A}_{\mu,t},$$

which is the inclusion we wish to prove. Here $\mathrm{Ran}(T)$ denotes the range of an operator T.

We next note that the inclusion map is equal to $A_{\mu,t} M_t (C_{\mu,t})^{-1}$, since this acts as the identity on its domain $\mathcal{C}_{\mu,t}$ and has codomain $\mathcal{A}_{\mu,t}$. Therefore the inclusion map $\iota : \mathcal{C}_{\mu,t} \hookrightarrow \mathcal{A}_{\mu,t}$, being the composition of two bounded operators, is bounded. Its operator norm satisfies

$$||\iota|| = ||A_{\mu,t} M_t (C_{\mu,t})^{-1}|| \leq ||A_{\mu,t}|| \, ||M_t|| \, ||(C_{\mu,t})^{-1}|| = 1,$$

exactly what one requires of an inclusion for it to be contractive. □

Remark 2.2. Using the different normalizations in [6] this inclusion is bounded, but not contractive.

Even though equation (2.1) immediately implies for $z \in \mathbb{C}^N$ and $q \in \mathbb{R}^N$ that

$$A_{\mu,t}(z, q) = \frac{C_{\mu,t}(z, q)}{A_{\mu,t}(0, q)}, \tag{2.4}$$

this factorization of $A_{\mu,t}(z, q)$ is not very useful, since $(A_{\mu,t}(0, q))^{-1} = e^{q^2/4t}$ is not a bounded function of q. So we are not able to prove the opposite inclusion $\mathcal{C}_{\mu,t} \supset \mathcal{A}_{\mu,t}$ using (2.4). Actually, we have the following.

Theorem 2.3. *The complementary set $\mathcal{A}_{\mu,t} \setminus \mathcal{C}_{\mu,t}$ is non-empty, that is, there exists $f \in \mathcal{A}_{\mu,t}$ such that $f \notin \mathcal{C}_{\mu,t}$.*

Proof. It is known (see [21]) that $\mathcal{C}_{\mu,t}$ is a reproducing kernel Hilbert space with reproducing kernel function

$$L_{\mu,t}(z, w) = c\rho_{\mu,2t}(z^*, w)$$

for all $z, w \in \mathbb{C}^N$, where the value of the constant $c > 0$ is not important for us now. So any $f \in \mathcal{C}_{\mu,t}$ satisfies the usual pointwise bound for a reproducing kernel Hilbert space, namely

$$|f(z)| \leq (L_{\mu,t}(z, z))^{1/2} \, ||f||_{\mathcal{C}_{\mu,t}}$$

for all $z \in \mathbb{C}^N$. Next we use the definition of the Dunkl heat kernel to calculate

$$L_{\mu,t}(z,z) = c\rho_{\mu,2t}(z^*,z) = ce^{-((z^*)^2+z^2)/4t} E_\mu\left(\frac{z^*}{(2t)^{1/2}}, \frac{z}{(2t)^{1/2}}\right).$$

We write $z = x + iy$ with $x, y \in \mathbb{R}^N$ and so get $(z^*)^2 + z^2 = 2\,\mathrm{Re}(z^2) = 2(x^2 - y^2)$. We then use the estimates (see [14])

$$0 \le E_\mu\left(\frac{z^*}{(2t)^{1/2}}, \frac{z}{(2t)^{1/2}}\right) \le e^{||z||^2/2t} = e^{(x^2+y^2)/2t}$$

to conclude that

$$|f(z)| \le c^{1/2} e^{-(x^2-y^2)/4t} e^{(x^2+y^2)/4t} \, ||f||_{\mathcal{C}_{\mu,t}} = c^{1/2} e^{y^2/2t} \, ||f||_{\mathcal{C}_{\mu,t}}.$$

In particular, it follows that f restricted to $\mathbb{R}^N$ is a bounded function for every $f \in \mathcal{C}_{\mu,t}$. This implies that the only holomorphic polynomials in $\mathcal{C}_{\mu,t}$ are the constants. But we know that $p \in \mathcal{A}_{\mu,t}$ for *all* holomorphic polynomials p. (See [2].) And this shows that $\mathcal{A}_{\mu,t} \setminus \mathcal{C}_{\mu,t}$ is non-empty. □

However, we shall see later on in the next section that the relation between the Version A and the Version C Segal-Bargmann spaces is different in the case of compact Lie groups. But first, we present some relations in the Coxeter context among Versions A, B and C of the Segal-Bargmann transform and the Dunkl heat kernel $\rho_{\mu,t}$ restricted to $\{0\} \times \mathbb{R}^N$, that is, $\sigma_{\mu,t}(q) = \rho_{\mu,t}(0,q)$ for $q \in \mathbb{R}^N$.

Theorem 2.4. *For $q \in \mathbb{R}^N$ and $z \in \mathbb{C}^N$ we have the identities*

$$B_{\mu,t}(z,q) = A_{\mu,t}(z,q) \,/\, A_{\mu,t}(0,q) \tag{2.5}$$

$$\rho_{\mu,t}(z,q) = C_{\mu,t}(z,q) = A_{\mu,t}(0,q) A_{\mu,t}(z,q) \tag{2.6}$$

$$\sigma_{\mu,t}(q) = \rho_{\mu,t}(0,q) = (A_{\mu,t}(0,q))^2 \tag{2.7}$$

$$C_{\mu,t}(2z,q) = A_{\mu,2t}(2z,0) A_{\mu,t/2}(z,q) \tag{2.8}$$

which tell us that we can obtain Versions B and C as well as the heat kernel $\sigma_{\mu,t}$ on $\mathbb{R}^N$ from Version A. We also have the identities

$$A_{\mu,t}(z,q) = C_{\mu,t}(z,q) \,/\, (C_{\mu,t}(0,q))^{1/2} \tag{2.9}$$

$$B_{\mu,t}(z,q) = C_{\mu,t}(z,q) \,/\, C_{\mu,t}(0,q) \tag{2.10}$$

$$\sigma_{\mu,t}(q) = \rho_{\mu,t}(0,q) = C_{\mu,t}(0,q) \tag{2.11}$$

which tell us that we also can get Versions A and B and the heat kernel $\sigma_{\mu,t}$ on $\mathbb{R}^N$ from Version C.

Remark 2.5. It is curious that in the present Coxeter context Version A determines Version C via the two distinct identities (2.6) and (2.8). We do not pretend to have any deeper understanding of this fact. We will see that in the compact Lie group context only (2.6) has a valid analogue, while the analogue of (2.8) is false at least for the Lie group $SU(2)$.

The identity (2.8) tells us that the isometry property of either of the transforms $A_{\mu,t}$ and $C_{\mu,t}$ can be deduced from the isometry property of the other. This is because the first factor on the right-hand side of (2.8) does not depend on q. So, for example, this factor can be factored out of the integral defining the C version of the Segal-Bargmann transform, leaving under the integral a kernel function for the A version of the Segal-Bargmann transform, although with a different "time" parameter. I thank the anonymous referee for bringing this to my attention. But one can easily fall into a trap by thinking that the factor $A_{\mu,2t}(2z, 0)$ in front of the integral sign must necessarily be involved in a "change of measure" argument. The relevant point here is that neither of the co-domain Hilbert spaces for the transforms $A_{\mu,t}$ and $C_{\mu,t}$ has an inner product defined by a measure using an L^2 type formula, but rather by a reproducing kernel function. It remains an open problem in this field of research whether these Hilbert space inner products can be so represented by measures. In short, if such measures exist, they are not known at the present time. (Our conjecture is that they do not exist in general. However, see [18] to see how this may be possible with more than one measure.) Of course, a "change of measure" argument makes no sense without measures. Nonetheless, there is a "change of reproducing kernel" theory, though it seems not to be so well known. This theory can be applied in the current context to show that the isometry of either one of these versions of the Segal-Bargmann transform implies the isometry of the other version, although to describe these implications as being "immediate" would be an exaggeration. The details of this argument would lead us too far afield and so are left to the interested reader. Since the analogue of (2.8) does not hold in general in the compact Lie group case, the remarks of this paragraph have no general analogue in that case.

Proof. For (2.5) we use (1.3) and (1.1) to compute

$$B_{\mu,t}(z,q) = \frac{\rho_{\mu,t}(z,q)}{\rho_{\mu,t}(0,q)} = e^{-z^2/2t} E_\mu\left(\frac{z}{t^{1/2}}, \frac{q}{t^{1/2}}\right). \tag{2.12}$$

We recall that $A_{\mu,t}(0,q) = e^{-q^2/4t}$ which together with (1.2) implies that

$$\frac{A_{\mu,t}(z,q)}{A_{\mu,t}(0,q)} = e^{-z^2/2t} E_\mu\left(\frac{z}{t^{1/2}}, \frac{q}{t^{1/2}}\right). \tag{2.13}$$

Then equations (2.12) and (2.13) imply (2.5).

Next we note that (2.6) is exactly (2.1), first proved in [20]. To obtain (2.7) we put $z = 0$ into (2.6).

We first proved (2.8) in [20]. This is a generalization of equation (A.18) in Hall's paper [5], which corresponds to the case $\mu \equiv 0$ of (2.8). This identity seems to be related to the fact that the underlying Riemannian manifolds $\mathbb{R}^N$ and $\mathbb{C}^N$ are flat Euclidean spaces.

To prove (2.9) we calculate that

$$\begin{aligned}\frac{C_{\mu,t}(z,q)}{(C_{\mu,t}(0,q))^{1/2}} &= \frac{\rho_{\mu,t}(z,q)}{(\rho_{\mu,t}(0,q))^{1/2}} = \frac{\rho_{\mu,t}(z,q)}{(e^{-q^2/2t})^{1/2}} = e^{q^2/4t}\rho_{\mu,t}(z,q)\\ &= e^{q^2/4t} e^{-(z^2+q^2)/2t} E_\mu\left(\frac{z}{t^{1/2}}, \frac{q}{t^{1/2}}\right) = e^{-z^2/2t - q^2/4t} E_\mu\left(\frac{z}{t^{1/2}}, \frac{q}{t^{1/2}}\right) = A_{\mu,t}(z,q).\end{aligned}$$

For (2.10) we merely note that by definition we have $C_{\mu,t}(z,q) = \rho_{\mu,t}(z,q)$, and then we apply (2.12). And finally we remark that (2.11) is just a special case of $C_{\mu,t}(z,q) = \rho_{\mu,t}(z,q)$. □

Remark 2.6. Some of the results of Theorem 2.4, such as (2.11), are well known, while (2.6) is a relatively recent result. We have presented all these identities together to emphasize the exact relations among the three versions in the Coxeter case. We will then use all this as motivation for the results in the next section.

3. Lie group case

We now examine the corresponding case introduced by Hall in [5] for a compact, connected (real) Lie group K. We first review some material from [5] and refer the reader to that paper for more details. Now K has a complexification, which is a complex Lie group G. Among other things, K is a Lie subgroup of G. For every $t > 0$ there is a heat kernel $\rho_t : K \to (0, \infty)$, which has a unique holomorphic extension (also denoted as ρ_t) with $\rho_t : G \to \mathbb{C}$. We continue to consider $t > 0$ in the following as Planck's constant and as having a fixed value.

The integral kernel function for Version A is defined by

$$A_t(g,x) := \frac{\rho_t(x^{-1}g)}{(\rho_t(x))^{1/2}}$$

for $g \in G$ and $x \in K$. Here $x^{-1}g$ is in G (but not necessarily in K) and so the ρ_t in $\rho_t(x^{-1}g)$ refers to the holomorphic extension. The corresponding Version A Segal-Bargmann transform is then defined by

$$A_t\psi(g) := \int_K \mathrm{d}_H x\, A_t(g,x)\psi(x)$$

for all $\psi \in L^2(K, d_H x)$ and all $g \in G$, where $d_H x$ is the normalized Haar measure of the compact group K. Theorem 1 in [5] states that $A_t : L^2(K, d_H x) \to \mathcal{H}L^2(G, \mu_t)$ is a unitary isomorphism, where μ_t is a heat kernel measure on G (and not to be confused with our notation μ for the multiplicity function) and $\mathcal{H}L^2(G, \mu_t)$ denotes the closed subspace of holomorphic functions in $L^2(G, \mu_t)$.

Theorem 2 in [5] states that $C_t : L^2(K, d_H x) \to \mathcal{H}L^2(G, \nu_t)$ is a unitary isomorphism, where ν_t is the measure on G that we get by averaging μ_t over the left action of K on G, using the fact that K is a subgroup of G. Of course, $\mathcal{H}L^2(G, \nu_t)$ denotes the closed subspace of holomorphic functions in $L^2(G, \nu_t)$. The definition of the Version C Segal-Bargmann transform is

$$C_t\psi(g) := \int_K \mathrm{d}_H x\, C_t(g,x)\psi(x)$$

for all $\psi \in L^2(K, d_H x)$ and all $g \in G$, where the kernel function is defined by

$$C_t(g,x) := \rho_t(x^{-1}g)$$

for $g \in G$ and $x \in K$.

However, for the B Version we are using our convention (see [20]) that

$$B_t(g,x) := \rho_t(x^{-1}g) \,/\, \rho_t(x)$$

for $g \in G$ and $x \in K$, which differs from the convention in [5]. In our convention a kernel function of two variables $T(x,y)$ determines an associated integral kernel transform T by

$$Tf(x) := \int_Y \mathrm{d}\nu(y)\, T(x,y) f(y),$$

where (Y,ν) is a measure space and f is in a space associated with the measure ν, say in $L^p(Y,\nu)$ for some p. Note that we use the same symbol for the kernel function as well as for its associated operator. This is a common abuse of notation.

So our definition of Version B Segal-Bargmann transform reads

$$B_t\phi(g) := \int_K \mathrm{d}\rho_t(x)\, B_t(g,x)\phi(x),$$

where $\mathrm{d}\rho_t(x) := \rho_t(x)\,\mathrm{d}_H x$, for all $g \in G$ and $\phi \in L^2(K,\rho_t)$. This is equivalent to the definition given in [5].

The reader should note the analogy between this material from [5] and our corresponding material in [20], which was motivated by [5]. In contrast, in this paper our results in the Coxeter context given in the previous section will be used to motivate the study of analogous results in the Lie group case.

Another analogy with the Coxeter case concerns the heat kernel. In the Lie group context the heat kernel $\rho_t : K \to (0,\infty)$ determines two more kernels. But first for each $x \in K$ and $f : K \to \mathbb{C}$ we define the translation of f by x to be $(\mathcal{T}_x f)(y) := f(x^{-1}y)$ for all $y \in K$. This definition has the virtue that $\mathcal{T}_{x_1}\mathcal{T}_{x_2} = \mathcal{T}_{x_1x_2}$. Then we define the two-variable heat kernel $\rho_t : K \times K \to (0,\infty)$ (using the same notation ρ_t for this function) for $x, y \in K$ by

$$\rho_t(x,y) := (\mathcal{T}_x\rho_t)(y).$$

This kernel in turn has an analytic continuation $\rho_t : G \times G \to \mathbb{C}$ (denoted again with the same notation), which is used in the definitions of the kernel functions for all three versions of the Segal-Bargmann transform in the Lie group context.

We would also like to note that there seems to be a limit as to how far one can find analogies between the Coxeter case and the Lie group case. For example, as noted above, in the Lie group case the heat kernel measure of G plays an important role in defining the spaces of holomorphic functions on G. However, even when $N = 1$, the definition of the holomorphic function spaces in the Coxeter case uses in general more than one measure. (See [18].)

We now are about ready to state our result for Lie groups. But first we remark that e denotes the identity element in $K \subset G$.

Theorem 3.1. *Let K be a compact, connected Lie group, and let G denote its complexification. Then we have for all $g \in G$ and $x \in K$ the identities*

$$B_t(g,x) = A_t(g,x) \,/\, A_t(e,x) \tag{3.1}$$

$$C_t(g,x) = A_t(e,x)A_t(g,x) \tag{3.2}$$

$$\rho_t(x) = (A_t(e,x))^2, \tag{3.3}$$

which tell us that from Version A we can obtain Versions B and C as well as the heat kernel ρ_t on K (and hence implicitly its analytic extension to G). We also have the identities

$$A_t(g,x) = C_t(g,x) \,/\, (C_t(e,x))^{1/2} \tag{3.4}$$

$$B_t(g,x) = C_t(g,x) \,/\, C_t(e,x) \tag{3.5}$$

$$\rho_t(x) = C_t(e,x), \tag{3.6}$$

which tell us that we can also get Versions A and B and the heat kernel ρ_t on K from Version C. Finally, we have that

$$\mathcal{H}L^2(G,\mu_t) = \mathcal{H}L^2(G,\nu_t). \tag{3.7}$$

Remark 3.2. All of the identities in Theorem 2.4 have an analogue here except for equation (2.8). The identity (3.2), which we believe to be new even though it is quite elementary, shows that in this Lie group context the Segal-Bargmann transform for Version A determines that for Version C. While it remains true that Version A and Version C are different (as in the Coxeter context), there is an *essential* relation between them and, indeed, a relation that also holds analogously in the Coxeter context.

Proof. Let $g \in G$ and $x \in K$ be arbitrary in this proof. For (3.2) we simply use the definitions and $\rho_t(x^{-1}) = \rho_t(x)$ (see [5], p. 108) to evaluate

$$A_t(e,x)A_t(g,x) = \frac{\rho_t(x^{-1}e)}{(\rho_t(x))^{1/2}} \cdot \frac{\rho_t(x^{-1}g)}{(\rho_t(x))^{1/2}} = \rho_t(x^{-1}g) = C_t(g,x).$$

As in the Coxeter case, (3.2) immediately implies a bounded inclusion, namely

$$\mathrm{Ran}\, C_t \subset \mathrm{Ran}\, A_t$$

since $A_t(e,x)$ as a function of $x \in K$ is bounded, K being compact. By (3.2)

$$A_t(g,x) = \frac{C_t(g,x)}{A_t(e,x)}, \tag{3.8}$$

which *is* useful unlike (2.4). This is so since the denominator satisfies

$$A_t(e,x) = \frac{\rho_t(x^{-1}e)}{(\rho_t(x))^{1/2}} = (\rho_t(x))^{1/2} > 0, \tag{3.9}$$

and so is bounded from below away from 0, since x varies in K compact. (For the inequality $\rho_t(x) > 0$, see [5].) Given this fact, equation (3.8) now implies that

$$\mathrm{Ran}\, A_t \subset \mathrm{Ran}\, C_t$$

is a bounded inclusion. Together with the previous inclusion, this shows that

$$\operatorname{Ran} A_t = \operatorname{Ran} C_t$$

which completes the first part of the proof of (3.7).

However to show the identity (3.7) requires identifying the ranges of A_t and C_t to be equal to the respective Hilbert spaces $\mathcal{H}L^2(G,\mu_t)$ and $\mathcal{H}L^2(G,\nu_t)$. That is, one must show that A_t and C_t are onto their respective Hilbert spaces and with proofs that do not depend on the identity (3.7). Unfortunately, the first paper to show the surjectivity of these transforms (which is [5]) does use this property. (See Lemma 11 in [5].) However, restriction principles provide another approach to this theory, and this approach leads to quick, rather elementary proofs of the surjectivity (as well as the injectivity) of these two transforms. This approach is sketched in [12] for both Versions A and C, while it is presented in the recent paper [8] in more detail, but only for the C-version. And this completes the proof of (3.7). The author thanks the anonymous referee for pointing out the importance of the comments in this paragraph.

The identity (3.3) follows immediately from (3.9). Next, using the definitions of $A_t(g,x)$ and $B_t(g,x)$ as well as (3.9), we calculate

$$\frac{A_t(g,x)}{A_t(e,x)} = \frac{\rho_t(x^{-1}g)}{(\rho_t(x))^{1/2}} \cdot \frac{1}{(\rho_t(x))^{1/2}} = \frac{\rho_t(x^{-1}g)}{\rho_t(x)} = B_t(g,x),$$

which is exactly (3.1).

To show (3.4) we simply note that

$$\frac{C_t(g,x)}{(C_t(e,x))^{1/2}} = \frac{\rho_t(x^{-1}g)}{(\rho_t(x^{-1}))^{1/2}} = \frac{\rho_t(x^{-1}g)}{(\rho_t(x))^{1/2}} = A_t(g,x).$$

For (3.5) we use $C_t(e,x) = \rho_t(x^{-1}) = \rho_t(x)$ and definitions to get

$$\frac{C_t(g,x)}{C_t(e,x)} = \frac{\rho_t(x^{-1}g)}{\rho_t(x)} = B_t(g,x),$$

which proves (3.5). We have also just proved (3.6), thereby finishing the proof. □

Remark 3.3. Of course, the heat kernel on K determines all three versions of the Segal-Bargmann transform, this being precisely a major theme of Hall's paper [5]. The previous theorem shows that each of the Versions A and C determines the remaining two versions as well as determining the heat kernel of K. It seems that the Version B does not determine these other structures.

The identities (3.1)–(3.6) are all easy to prove and so it would be surprising if they are all new. In fact, some of them clearly are not new, such as (3.6). However, the identity (3.2) does seem to be new in this context. But its consequence (3.7) was already known, since that follows from the stronger result

$$L^2(G,\mu_t) = L^2(G,\nu_t),$$

which in turn follows immediately from Lemma 11 in [5] (p. 124). However, (3.2) together with the powerful, yet elementary restriction principles gives us a new, conceptually simple proof of (3.7).

The importance of (3.2) is that it tells us Version C of the Segal-Bargmann transform is determined by Version A of the Segal-Bargmann transform, where we understand that an integral kernel transform is “equivalent” to its integral kernel function. And though (3.2) could have been proved in [5], it does not appear there. This result, though extremely simple, was a complete surprise to this author, who had interpreted the last sentence in [5] as implying the impossibility of any relation of the sort. In short, the importance of the identity (3.2) is that it now clarifies this issue. And again, as far as this author is aware, this is the first time that this explicit relation between Versions A and C in the context of Lie groups has appeared in the published literature.

We also wish to note that the inclusions and equalities of spaces given in this section are *as sets* and not as Hilbert spaces. This is because the inner products do not coincide.

4. An interesting counterexample: $SU(2)$

As we have already remarked, the identity (2.8) in the Coxeter context does not have an analogue in the Lie group context. We now construct a counterexample to show that the analogous equation is false in general. We present this in detail, since we find it to be a rather nice exercise in finding an elegant and useful formula for the heat kernels of $SU(2)$ and $SL(2;\mathbb{C})$. Much of the material in this section is classical. We have chosen to start by following the presentation and notation given in Chapter 7 of [9].

We now consider the case of the compact Lie group $SU(2)$ of 2×2 complex matrices A which are unitary (that is, $A^*A = I$) and have determinant one. We use notation for group elements and the identity matrix (namely, I) that is standard for matrix groups. By the spectral theorem for normal operators, there exists a unitary matrix B which diagonalizes a given $A \in SU(2)$, that is $B^{-1}AB$ is diagonal. By taking $C = B/(\det B)^{1/2} \in SU(2)$, where $(\det B)^{1/2}$ is one of the square roots of $\det B$, we have that A is conjugate in $SU(2)$ to

$$C^{-1}AC = \begin{pmatrix} e^{\mathrm{i}\tau/2} & 0 \\ 0 & e^{-\mathrm{i}\tau/2} \end{pmatrix} \tag{4.1}$$

for some real number $\tau \in [0, 4\pi)$, since the eigenvalues α, β of A (and also of its diagonalization) satisfy $|\alpha| = |\beta| = 1$ and $\alpha\beta = 1$. The condition on τ is not too restrictive since it still allows the $(1, 1)$ entry (and also the $(2, 2)$ entry) in the matrix (4.1) to achieve any value on the unit circle. But by conjugating formula (4.1) by the matrix

$$\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \in SU(2),$$

which interchanges the eigenvalues on the diagonal of formula (4.1), we see that the matrices in (4.1) with parameter $\tau \in [2\pi, 4\pi)$ are conjugate in $SU(2)$ to the matrices with parameter $\tau' \in (0, 2\pi]$, where $\tau' = 4\pi - \tau$.

Moreover, for $\tau_1, \tau_2 \in [0, 2\pi]$ with $\tau_1 \neq \tau_2$, the corresponding matrices are not conjugate, since they have different sets of eigenvalues. So, by taking $\tau \in [0, 2\pi]$ in (4.1), we get a family of matrices which contains exactly one representative of each conjugacy class in $SU(2)$, that is, the value of τ in $[0, 2\pi]$ is now uniquely determined for each $A \in SU(2)$.

Even though we could label the irreducible unitary representations of $SU(2)$ by their dimensions, it is conventional to label them by the non-negative half integers u (those non-negative real numbers u such that $2u$ is an integer) such that $2u + 1$ is the dimension of the representation. If ϕ_u denotes the associated irreducible representation, then we have that $\phi_u(A)$ is a $(2u+1) \times (2u+1)$ unitary matrix for every $A \in SU(2)$. The corresponding character $\chi_u = \mathrm{Tr} \circ \phi_u$ (where Tr is the trace of a matrix) is a complex-valued function that is constant on each conjugacy class of $SU(2)$. So, $\chi_u(A)$ is a function of $\tau \in [0, 2\pi]$ only. Actually, this function can be calculated explicitly for $A \in SU(2)$ as

$$\chi_u(A) = \sum_{s=-u}^{u} e^{is\tau} = \frac{\sin(u + 1/2)\tau}{\sin(\tau/2)},$$

where these formulas can be found in [9], p. 232. The last formula results by summing the finite geometric series and simplifying. (The singularities in the last expression are removable and are understood as having been removed.) Note that the summation in the second expression is taken in unit steps, even in the case when u is not an integer. For example, when $u = 3/2$ the sum is over s equal to the four values $-3/2$, $-1/2$, $1/2$, $3/2$. In general, the sum contains $2u + 1$ terms.

We have used [9] as a guide for the discussion so far but now take a different tack, since we wish to write τ in terms of the matrix A. Note that any $A \in SU(2)$ can be written as

$$A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

with $a, b \in \mathbb{C}$ satisfying $|a|^2 + |b|^2 = 1$. Therefore we have that

$$\mathrm{Tr}(A) = a + a^* = 2\,\mathrm{Re}(a) \tag{4.2}$$

and by (4.1) that

$$\mathrm{Tr}(A) = \mathrm{Tr}(C^{-1}AC) = e^{\mathrm{i}\tau/2} + e^{-\mathrm{i}\tau/2} = 2\cos(\tau/2).$$

So we have $\mathrm{Re}(a) = \cos(\tau/2)$ or equivalently

$$\tau = 2\cos^{-1}(\mathrm{Re}(a)).$$

Here we are using the standard definition $\cos^{-1} : [-1, 1] \to [0, \pi]$. We note that $\mathrm{Re}(a) \in [-1, 1]$, since $|\,\mathrm{Re}(a)|^2 \leq |a|^2 \leq |a|^2 + |b|^2 = 1$. So this is in agreement with our earlier restriction that $\tau \in [0, 2\pi]$, that is, our choice for the branch of the inverse cosine is correct.

Returning to the character, we see that

$$\chi_u(A) = \frac{\sin(u+1/2)\tau}{\sin(\tau/2)} = \frac{\sin[(2u+1)\cos^{-1}(\mathrm{Re}(a))]}{\sin[\cos^{-1}(\mathrm{Re}(a))]},$$

which already expresses the character of $A \in SU(2)$ in terms of an entry of the matrix A, namely a, although the formula seems to leave something to be desired.

Now we recall the definition of the *Chebyshev polynomial of the second kind* (see [10]) for any $x \in (-1, 1)$ and integer $n \geq 0$ as

$$U_n(x) := \frac{\sin((n+1)\theta)}{\sin\theta} = \frac{\sin((n+1)\cos^{-1} x)}{\sin(\cos^{-1} x)},$$

where $x = \cos\theta$ or $\theta = \cos^{-1} x$. One verifies that this is a polynomial function of x in the interval $(-1, 1)$ and then extends the domain of definition of U_n to the entire complex plane by analytic continuation.

So, for $|\mathrm{Re}(a)| < 1$, we finally arrive at the rather simple expressions

$$\chi_u(A) = U_{2u}(\mathrm{Re}(a)) = U_{2u}\left(\frac{1}{2}\,\mathrm{Tr}(A)\right), \tag{4.3}$$

where the second equality comes from equation (4.2). (The case $|\mathrm{Re}(a)| = 1$ occurs if and only if $A = \pm I$. Then the proof of (4.3) follows by continuity.) Now these are more elegant ways of writing the character of $A \in SU(2)$ in terms of A itself. We must note here that the formula (4.3) is known, but apparently not that well appreciated. For example, Miller notes in [9], p. 233, that he is aware of this formula, but he does not present it since he considers it to be "not very enlightening." This is why we have presented and proved (4.3) here.

Next, according to equation (15) in [5], the heat kernel of the compact Lie group $SU(2)$ is given for $A \in SU(2)$ and $t > 0$ by

$$\rho_t(A) = \sum_u \dim(\phi_u) e^{-\lambda_u t/2}\, \chi_u(A),$$

where $\dim(\phi_u) = 2u + 1$ and λ_u is the *unique* eigenvalue of minus the Laplacian acting in the representation space, that is $\phi_u(-\Delta) = \lambda_u I$. We have $\lambda_u = u(u+1)$. (See [9].) So we have that

$$\begin{aligned}\rho_t(A) &= \sum_u (2u+1) e^{-u(u+1)t/2}\, U_{2u}\left(\frac{1}{2}\,\mathrm{Tr}(A)\right) \\ &= \sum_{n=0}^{\infty} (n+1) e^{-n(n+2)t/8}\, U_n\left(\frac{1}{2}\,\mathrm{Tr}(A)\right),\end{aligned} \tag{4.4}$$

where the first sum is over all non-negative half-integers u and the second is over all integers $n \geq 0$, where $n = 2u$. Hall proves in [5] that this series converges absolutely for all $t > 0$ and all $A \in SL(2;\mathbb{C})$. However, by using properties of the polynomials U_n, one can directly prove the absolute convergence of this series for all $t > 0$ and for *any* 2×2 complex matrix A, as we will show momentarily. Also, it is known that $\rho_t(A) > 0$ for all $t > 0$ and for all $A \in SU(2)$, but this is not obvious from formula (4.4), since U_n has n simple roots in $[-1, 1]$.

We now consider how to find the analytic continuation of the heat kernel ρ_t to the complexification of $SU(2)$, which also can be identified with the cotangent bundle of $SU(2)$. It turns out that the complexification of $SU(2)$ is $SL(2;\mathbb{C})$, the group of all 2×2 matrices with complex entries and determinant one. We will next show that the analytic continuation of ρ_t is given by the same formula (4.4) given above, but now for $A \in SL(2;\mathbb{C})$. Actually, we will procede by proving the uniform absolute convergence of the series in (4.4) on compact subsets of $M(2;\mathbb{C})$, the space of all 2×2 complex matrices. First, we need a lemma.

Lemma 4.1. *For all $z \in \mathbb{C}$ and every integer $k \geq 0$ we have this estimate for the Chebyshev polynomials U_k of the second kind:*

$$|U_k(z)| \leq (\,3 \max(1, |z|)\,)^k$$

Remark 4.2. This estimate is not optimal. Nor is it meant to be.

Proof. The proof is by induction on k. For $k = 0$ and $k = 1$ the estimate is easy enough, using $U_0(z) = 1$ and $U_1(z) = 2z$, and so is left to the reader. We now assume that $n \geq 1$ and that the estimate holds for $k = n$ and $k = n-1$. It remains for us to show the estimate for $k = n+1$. We will use the three term recursion relation for the Chebyshev polynomials of the second kind:

$$U_{n+1}(z) = 2zU_n(z) - U_{n-1}(z)$$

for $n \geq 1$. (See [3].) We first consider the case $|z| \geq 1$. Using the induction hypothesis we have that

$$\begin{aligned} |U_{n+1}(z)| &\leq 2|z|\,|U_n(z)| + |U_{n-1}(z)| \leq 2|z|\,3^n|z|^n + 3^{n-1}|z|^{n-1} \\ &\leq 2 \cdot 3^n|z|^{n+1} + 3^{n-1}|z|^{n+1} = (2 \cdot 3 + 1)\,3^{n-1}|z|^{n+1} \\ &\leq 3^2\,3^{n-1}|z|^{n+1} = 3^{n+1}|z|^{n+1}, \end{aligned}$$

which is the estimate for $n+1$ in this case. The case $|z| \leq 1$ is proved similarly. □

Theorem 4.3. *The series in* (4.4) *converges absolutely and uniformly on compact subsets of $M(2;\mathbb{C})$. Consequently, it defines a holomorphic function on the complex manifold $M(2;\mathbb{C}) \cong \mathbb{C}^4$.*

Proof. Consider a compact subset $S \subset M(2;\mathbb{C})$. Define

$$C_S := 3 \max\left(\,1, \sup_{A \in S} \frac{1}{2}|\operatorname{Tr}(A)|\,\right).$$

Then by the previous lemma we have that $|U_n(\frac{1}{2}\operatorname{Tr}(A))| \leq (C_S)^n$ for all $A \in S$. We use this and the root test to estimate for $A \in S$ as follows:

$$\sum_{n=0}^{\infty} \left|(n+1)e^{-n(n+2)t/8}\,U_n\left(\frac{1}{2}\operatorname{Tr}(A)\right)\right| \leq \sum_{n=0}^{\infty}(n+1)e^{-n(n+2)t/8}\,(C_S)^n < \infty.$$

The first statement of the theorem now follows from the Weierstrass M-test. Since $\mathrm{Tr} : M(2;\mathbb{C}) \to \mathbb{C}$ is clearly holomorphic and each U_n is a holomorphic polynomial, the partial sums of (4.4) are clearly holomorphic functions of $A \in M(2;\mathbb{C})$. So the second statement of the theorem follows immediately from the first statement. □

Remark 4.4. Since the inclusion mapping $SL(2;\mathbb{C}) \hookrightarrow M(2;\mathbb{C})$ is holomorphic, it follows that (4.4) for $A \in SL(2;\mathbb{C})$ gives the analytic continuation of ρ_t from $SU(2)$ to $SL(2;\mathbb{C})$. Theorem 4.3 follows from [5] (Prop. 1, p. 111) but only for $SL(2;\mathbb{C})$ instead of $M(2;\mathbb{C})$.

We now change notation by letting $X \in SU(2)$ and $G \in SL(2;\mathbb{C})$ denote generic elements in these two groups. Then the integral kernel for Version A of the Segal-Bargmann transform for $SU(2)$ is given by

$$A_t(G,X) = \frac{\rho_t(X^{-1}G)}{(\rho_t(X))^{1/2}},$$

while the kernel for Version C of the Segal-Bargmann transform for $SU(2)$ is

$$C_t(G,X) = \rho_t(X^{-1}G).$$

Finally, we now prove the main result of this section, namely that the identity analogous to (2.8) is not true for $SU(2)$.

Theorem 4.5. *The equation*

$$C_t(G^2,X) = A_{2t}(G^2,I)A_{t/2}(G,X) \tag{4.5}$$

is not identically true for all $X \in SU(2)$ and all $G \in SL(2;\mathbb{C})$.

Proof. Equation (4.5) is equivalent to

$$\rho_t(X^{-1}G^2) = \frac{\rho_{2t}(G^2)}{(\rho_{2t}(I))^{1/2}} \cdot \frac{\rho_{t/2}(X^{-1}G)}{(\rho_{t/2}(X))^{1/2}}. \tag{4.6}$$

Let us suppose that this is an identity and try to get a contradiction. First we calculate the heat kernel of two elements of $SU(2)$. For the identity matrix I we have that

$$\begin{aligned}\rho_t(I) &= \sum_{n=0}^{\infty}(n+1)e^{-n(n+2)t/8}\,U_n\left(\frac{1}{2}\,\mathrm{Tr}(I)\right)\\ &= \sum_{n=0}^{\infty}(n+1)e^{-n(n+2)t/8}\,U_n(1) = \sum_{n=0}^{\infty}(n+1)^2e^{-n(n+2)t/8},\end{aligned}$$

where we used that $U_n(1) = n+1$ and $\mathrm{Tr}(I) = 2$. For the next calculation we use $-I \in SU(2)$ and $U_n(-1) = (-1)^n(n+1)$. We then have

$$\begin{aligned}\rho_t(-I) &= \sum_{n=0}^{\infty}(n+1)e^{-n(n+2)t/8}\,U_n\left(\frac{1}{2}\,\mathrm{Tr}(-I)\right)\\ &= \sum_{n=0}^{\infty}(n+1)e^{-n(n+2)t/8}\,U_n(-1) = \sum_{n=0}^{\infty}(-1)^n(n+1)^2e^{-n(n+2)t/8}.\end{aligned}$$

This clearly implies that $\rho_t(-I) < \rho_t(I)$. We also have $0 < \rho_t(-I)$ by the strict positivity of the heat kernel on $SU(2)$.

Next in (4.6) we take $X = I$ and $G = I$ to get

$$\rho_t(I) = \frac{\rho_{2t}(I)}{(\rho_{2t}(I))^{1/2}} \cdot \frac{\rho_{t/2}(I)}{(\rho_{t/2}(I))^{1/2}}.$$

We also take $X = -I$ and $G = -I$ in (4.6) thereby obtaining

$$\rho_t(-I) = \frac{\rho_{2t}(I)}{(\rho_{2t}(I))^{1/2}} \cdot \frac{\rho_{t/2}(I)}{(\rho_{t/2}(-I))^{1/2}}.$$

Note that in these last two equations all the values of the heat kernel are strictly positive real numbers. So, it follows that

$$0 < \rho_t(-I)(\rho_{t/2}(-I))^{1/2} = \rho_t(I)(\rho_{t/2}(I))^{1/2}.$$

And this contradicts $0 < \rho_t(-I) < \rho_t(I)$, which holds for *all* $t > 0$. □

Remark 4.6. There are surely many other ways to show that (4.6) is not an identity. For example, one could use a computer assisted proof.

5. Concluding remarks

We feel that a major, new result of this note is embodied in (3.2), which tells us that Version C is determined by Version A in the Lie group context. Moreover, (3.4) gives us the reciprocal relation that Version A is determined by Version C in the Lie group context. However, it seems to be the case that the isometry properties of these two Versions are somehow "independent" properties, unlike the Coxeter case. Exactly how to express this idea in a mathematically rigorous way (and then prove it, of course!) remains a challenge. It was our study of Segal-Bargmann analysis in the Coxeter context which motivated us to find these results.

It is reasonable to conjecture that (4.5) is false for every compact, connected, non-abelian Lie group K. In the contrary case it would be interesting to know for which such K equation (4.5) is an identity.

The multitude of analogies between the Segal-Bargmann theory associated to a Coxeter group and the Segal-Bargmann theory for compact Lie groups strongly suggests that there is more here than mere analogy. An avenue for further research would be to find out if there is for example a new general theory which has these two theories as special cases. However, there is a difference, which we would like to note, between these two theories. The Lie group case is based on a Laplacian associated to the Lie group. And this is a differential operator. But the Coxeter case is based on the Dunkl Laplacian, which is a differential-difference operator.

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Operator Theory:
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Nonlinear Scattering in the Lamb System

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Abstract. The goal of this paper is to survey recent results on scattering in nonlinear conservative Lamb' s systems. A Lamb' s system is a wave equation coupled with an equation of motion of a particle of mass $m \geq 0$. We describe the long time asymptotics in a global energy norm for all finite energy solutions with $m = 0$ [6] and $m > 0$ [7]. Under certain conditions on the potential, each solution in an appropriate functional space decays, in a global energy norm as $t \to \pm\infty$, towards the sum of a *stationary state* and an *outgoing wave*. The outgoing waves correspond to the 'in' and 'out' asymptotic states. For $m > 0$, we define nonlinear wave operators corresponding to the ones introduced in [6] and obtain a necessary condition for the existence of the asymptotic states. For $m = 0$ we state a conjecture for the asymptotic completeness and verify this for some particular potentials.

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1. Introduction

Energy transfer within interconnected mechanical systems is important in many real world settings. When a finite- and an infinite-dimensional systems are coupled, energy is radiated from the former and absorbed by the latter. We call this process *radiation damping.*

In a conservative context, radiation damping can describe dissipation (*e.g.* friction, viscosity), where energy may dissipate from one form (such as mechanical) into another (such as heat) of a larger conservative system.

An early physical model of radiation damping was introduced by Horace Lamb [1]. In the Lamb model an oscillator coupled to a string describes the free vibrations of a nucleus in an extended medium. The oscillator transfers energy to the string by generating waves as it moves. In many linear and nonlinear partial differential equations, it is fruitful to view the dynamics in terms of "particle-like"

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components. A decomposition into these types of models leads to an equivalent description in terms of two coupled subsystems: the first is finite-dimensional and governs the "particle-like" or bound state part of the solution, while the second is infinite-dimensional and dispersive. Coupling terms are responsible for how the dynamics of "particles" influence the field and how the dispersive wave-field influences the particle dynamics.

Let us comment on related works. H. Lamb consider the linear case when $F(y) = -\omega^2 y$. The Lamb system with general nonlinear $F(y)$ and the oscillator mass $m > 0$ has been considered in [2] where the questions of irreversibility and non-recurrence were discussed. The convergence to stationary states for nonlinear Lamb system in local energy seminorms was proved in [4, 5] for the scalar wave equation with $m \geq 0$ and the compactly supported initial data, here the existence of a global attractor has been established for the first time. In [9] metastable regimes were studied for the stochastic Lamb system. The methods and results [4, 5] were applied and extended in [11] to stability and instability analysis in some linear systems of the Lamb type. A model of a particle coupled to a wave field is studied in [3]. The paper [12] concerns an application of Lax-Phillips scattering theory to linear models of the Lamb type and existence of dynamics for a class of the nonlinear systems. The long time asymptotics with a dispersive wave in global energy norms were proved first in [13] for 1D Schrödinger equation, and extended i) in [14]–[16] to 3D wave, Klein-Gordon and Schrödinger coupled to a particle, ii) in [17] to 1D nonlinear relativistic equation with piecewise constant potential of thew nonlinearity. However, all the results concern solutions sufficiently close to a solitary manifold. In [18]–[21] similarly asymptotic were proved for all finite energy solutions to 3D wave, Klein-Gordon and Maxwell equations coupled to a particle.

The paper is organized as follows. In Section 2 the problem is formulated along with an introduction of the phase space and stationary states. In Sections 3–4 we formulate the main results. In Section 5, we define nonlinear scattering operators, and obtain a necessary condition for the existence of the asymptotic states. In Section 6 we study the asymptotic completeness. At last, in Section 7 we present and solve the direct and inverse scattering problem for the Lamb system en the linear case if $m > 0$.

2. Problem formulation and description

We consider the nonlinear conservative Lamb system with mass $m \geq 0$ given by

$$\left\{\begin{array}{ll} \ddot{u}(x,t) = u''(x,t), \ x \in \mathbb{R} \setminus \{0\}, & \\ m\ddot{y}(t) = F(y(t)) + u'(0+,\, t) - u'(0-,\, t); & \\ \quad y(t) := \qquad\qquad u(0,t), & \end{array}\right| \ t \in \mathbb{R}. \tag{2.1}$$

Here $\dot{u} := \frac{\partial u}{\partial t}$, $u' := \frac{\partial u}{\partial x}$ and so on. The solutions $u(x,t)$ take the values in $\mathbb{R}^d$ with $d \geq 1$. Physically, the problem describes small crosswise oscillations of an infinity string stretched parallel to the x-axis; a particle of $m \geq 0$ is attached to the string at the point $x = 0$; $F(y)$ is an external (**nonlinear**) force field perpendicular to x-axis, the field subjects the particle. The Lamb system (2.1) is as an example

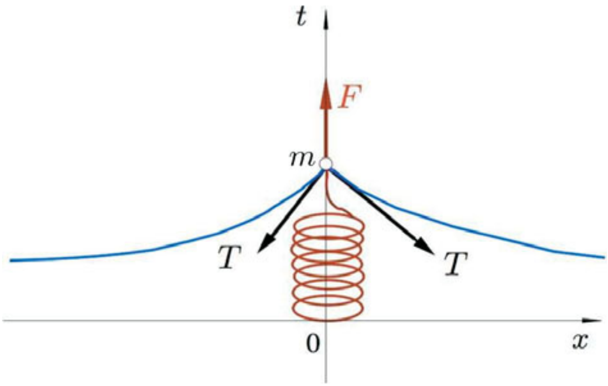

FIGURE 1. Lamb system ($d = 1$).

of simplest nontrivial time reversible conservative system allowing an effective analysis of various questions.

Lamb system (2.1) is *formally* equivalent to $\mathbb{R}^d$-valued one-dimensional nonlinear wave equation with the nonlinear term $\delta(x)F(u(x,t))$ concentrated at the single point $x = 0$:

$$\ddot{u}(x,t) = u''(x,t) + \delta(x)F(u(x,t)), \quad (x,t) \in \mathbb{R}^2, \quad \text{if} \quad m = 0, \tag{2.2}$$

$$\big(1 + m\delta(x)\big)\ddot{u}(x,t) = u''(x,t) + \delta(x)F(u(x,t)), \quad (x,t) \in \mathbb{R}^2, \quad \text{if} \quad m > 0. \tag{2.3}$$

The Cauchy problem consists in finding solutions $u(x,t)$, (in some functional space that we will specify shortly), of system (2.1) satisfying given initial conditions. These conditions differ in the cases $m = 0$ and $m > 0$:

$$u(x,t)|_{t=0} = u_0(x); \quad \dot{u}(x,t)|_{t=0} = v_0(x), \quad \text{if } m = 0, \tag{2.4}$$

and

$$u(x,t)|_{t=0} = u_0(x); \quad \dot{u}(x,t)|_{t=0} = v_0(x); \quad \dot{y}(t)|_{t=0} = p_0, \quad \text{if } m > 0. \tag{2.5}$$

Let us denote $Y(t) := (u(x,t), v(x,t))$ and $\mathsf{Y}(t) := (u(x,t), v(x,t), \dot{y}(t))$, where $v(x,t) := \dot{u}(x,t)$. Then the Cauchy problems (2.1), (2.4) (if $m = 0$) and (2.1), (2.5) (if $m > 0$) formally reads:

$$\begin{cases} \dot{Y}(t) = \mathbf{F}\big(Y(t)\big), & t \in \mathbb{R}, \\ Y(0) = Y_0, \end{cases} \tag{2.6}$$

where $Y_0 := (u_0, v_0)$, $\mathbf{F}\big(Y(t)\big) = \Big(v(x,t), u''(x,t) + \delta(x)F(u(x,t))\Big)$ if $m = 0$, and

$$\begin{cases} \dot{\mathsf{Y}}(t) = \mathsf{F}\big(\mathsf{Y}(t)\big), & t \in \mathbb{R}, \\ \mathsf{Y}(0) = \mathsf{Y}_0, \end{cases} \tag{2.7}$$

where

$$\mathsf{Y}_0 := (u_0, v_0, p_0), \quad \mathsf{F}\big(\mathsf{Y}(t)\big) = \Big(v(x,t), u''(x,t), F(u(x,t)) + u'(0+,t) - u'(0-,t)\Big)$$

if $m > 0$.

2.1. Notations and definitions

Let us introduce a phase space $\mathcal{E}^0$ of finite energy states for the system (2.1) with $m = 0$ and the corresponding space $\mathcal{E}$ if $m > 0$. Let $||\cdot||$ and $||\cdot||_R$ be the norms in the Hilbert spaces $L^2(\mathbb{R};\mathbb{R}^d)$ and $L^2(I_R;\mathbb{R}^d)$, respectively, where $I_R := (-R, R) \subset \mathbb{R}$, $R > 0$, generated by the scalar product:

$$\begin{aligned}\big\langle f(x), g(x)\big\rangle_{L^2(\mathbb{R},\mathbb{R}^d)} &:= \int\limits_{\mathbb{R}} f(x)\cdot g(x)dx \\ &= \int\limits_{\mathbb{R}} \big(f_1(x)g_1(x) + \cdots + f_d(x)g_d(x)\big)dx.\end{aligned}$$

Similarly we define $\big\langle f(x), g(x)\big\rangle_{L^2(I_R,\mathbb{R}^d)}$. And $|a| := \sqrt{\langle a, a\rangle}$, $a \in \mathbb{R}^d$.

Definition 2.1. i) $\mathcal{E}^0$ ($\mathcal{E}$) is the Hilbert space of the pairs $\big(u(x), v(x)\big) \in C(\mathbb{R};\mathbb{R}^d)$ $\oplus L^2(\mathbb{R};\mathbb{R}^d)$ (triples $\big(u(x), v(x), p\big) \in C(\mathbb{R};\mathbb{R}^d) \oplus L^2(\mathbb{R};\mathbb{R}^d) \oplus \mathbb{R}^d$) with $u'(x) \in L^2(\mathbb{R};\mathbb{R}^d)$ and the **global energy norms**

$$||(u,v)||_{\mathcal{E}^0} := ||u'|| + |u(0)| + ||v||, \qquad \text{if } m = 0, \tag{2.8}$$

$$||(u,v,p)||_{\mathcal{E}} := ||u'|| + |u(0)| + ||v|| + |p|, \quad \text{if } m > 0. \tag{2.9}$$

ii) $\mathcal{E}^0_F$ ($\mathcal{E}_F$) is the space $\mathcal{E}^0$ ($\mathcal{E}$) endowed with the topology defined by the **local energy seminorms**

$$||(u,v,p)||_{\mathcal{E}^0,R} := ||u'||_R + |u(0)| + ||v||_R, \qquad R > 0, \quad \text{if } m = 0, \tag{2.10}$$

$$||(u,v,p)||_{\mathcal{E},R} := ||u'||_R + |u(0)| + ||v||_R + |p|, \quad R > 0, \quad \text{if } m > 0. \tag{2.11}$$

3. Existence of dynamics

We assume that the nonlinear force field F has a real potential $V\colon \mathbb{R}^d \to \mathbb{R}$, $V \in C^2$:

$$F(u) \in C^2(\mathbb{R}^d;\mathbb{R}^d), \quad F(u) := -\nabla_u V(u) \;\text{ and }\; V(u) \xrightarrow[|u|\to\infty]{} +\infty \tag{3.1}$$

Then (2.1) is a formally Hamiltonian system with the phase space $\mathcal{E}^0$ (or $\mathcal{E}$) and the Hamiltonian functional

$$\mathcal{H}(u,v) = \frac{1}{2}\int_{\mathbb{R}} \Big[|v(x)|^2 + |u'(x)|^2\Big]dx + m\frac{|p|^2}{2} + V(u(0)), \tag{3.2}$$

for $(u,v) \in \mathcal{E}^0$, if $m = 0$ or $(u,v,p) \in \mathcal{E}$ if $m > 0$. Dynamics of Lamb system is established by the propositions:

Proposition 3.1. [6] *Let conditions* (3.1) *hold,* $m = 0$ *and* $d \geq 1$*. Then,*

1. *For every* $Y_0 \in \mathcal{E}^0$ *the Cauchy problem* (2.6) *admits a unique solution* $Y(t) \in C(\mathbb{R};\mathcal{E}^0)$.
2. *The map* $U(t) : Y_0 \mapsto Y(t)$ *is continuous in* $\mathcal{E}^0$ *and* $\mathcal{E}^0_F$.
3. *The energy is conserved*

$$\mathcal{H}(Y(t)) = \text{const}, \quad t \in \mathbb{R}.$$

4. *The a priori bound holds*

$$\sup_{t\in\mathbb{R}} ||Y(t)||_{\mathcal{E}^0} < \infty.$$

Proposition 3.2. [7] *Let conditions* (3.1) *hold,* $m > 0$ *and* $d \geq 1$*. Then,*

1. *For every* $\mathsf{Y}_0 \in \mathcal{E}$ *the Cauchy problem* (2.7) *admits a unique solution* $\mathsf{Y}(t) \in C(\mathbb{R};\mathcal{E})$.
2. *The map* $\mathsf{U}(t) : \mathsf{Y}_0 \mapsto \mathsf{Y}(t)$ *is continuous in* $\mathcal{E}$ *and* $\mathcal{E}_F$.
3. *The energy is conserved*

$$\mathcal{H}(\mathsf{Y}(t)) = \text{const}, \quad t \in \mathbb{R}.$$

4. *The a priori bound holds*

$$\sup_{t\in\mathbb{R}} ||\mathsf{Y}(t)||_{\mathcal{E}} < \infty.$$

Sketch of the proof. (Propositions 3.2 and 3.1, 1.) We consider $t > 0$, the case $t < 0$ its handled similar. First, the D' Alembert representation

$$u(x,t) = \frac{u_0(x-t) + u_0(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} v_0(\chi)d\chi, \quad |x| \geq t > 0, \tag{3.3}$$

implies the unique solution $u(x,t)$ in the region $|x| \geq |t|$. To prove the existence and uniqueness in the region $|x| < |t|$, we derive a nonlinear ordinary differential equation for $y(t) = u(0,t)$ from the second equation of (2.1). The contraction mapping principle implies the existence of a local solution $y(t)$ from the Cauchy problem for the **"reduced equation"**

$$\begin{cases} m\ddot{y}(t) = F(y(t)) - 2\dot{y}(t) + 2\dot{w}_{in}(t), & t > 0, \\ y(0) = u_0(0), \quad \dot{y}(0) = p_0, \end{cases} \quad \text{if } m > 0, \tag{3.4}$$

where $w_{in}(t) := g_+(t) + f_-(-t)$, for $t > 0$ is the *"incident wave"* and $f_\pm(z), g_\pm(z)$ for $\pm z > 0$ are defined by the well-known D' Alembert formulas (see [4]). The existence of the global solution and its continuity in $\overline{\mathbb{R}_+}$ follows from **a priori bounds**:

$$\sup_{t>0}|y(t)| + \int_0^\infty |\dot{y}(t)|^2 dt \le B < \infty, \quad \text{if } m = 0, \tag{3.5}$$

and

$$\sup_{t>0}|y(t)| + \sup_{t>0}|\dot{y}(t)| + \int_0^\infty |\dot{y}(t)|^2 dt \le \mathsf{B} < \infty, \quad \text{if } m > 0, \tag{3.6}$$

where B and B is bounded for $||(u_0, v_0)||_{\mathcal{E}^0}$ and $||(u_0, v_0, p_0)||_{\mathcal{E}}$ bounded respectively. Hence

$$u(x,t) = \begin{cases} y(t-x) + g_+(x+t) - g_+(t-x), & 0 < x < t \\ y(t+x) + f_-(x-t) - f_-(-x-t), & -t < x < 0 \end{cases} \Bigg| \; t > 0, \tag{3.7}$$

represents to the solution in the region $|x| < t$, with $t > 0$. Note that this formula contains only the incident waves. These arguments imply that the Cauchy problem (2.6) (or (2.7)) (see [4, 6, 7]) admits a unique solution $Y(t) = (u(x,t), v(x,t)) \in C(\mathbb{R}; \mathcal{E}^0)$ (or $\mathsf{Y}(t) = (u(x,t), v(x,y), \dot{y}(t)) \in C(\mathbb{R}; \mathcal{E})$) for any $Y_0 \in \mathcal{E}^0$ (or $\mathsf{Y}_0 \in \mathcal{E}$), where $u(x,t)$ is defined by (3.3) and (3.7). □

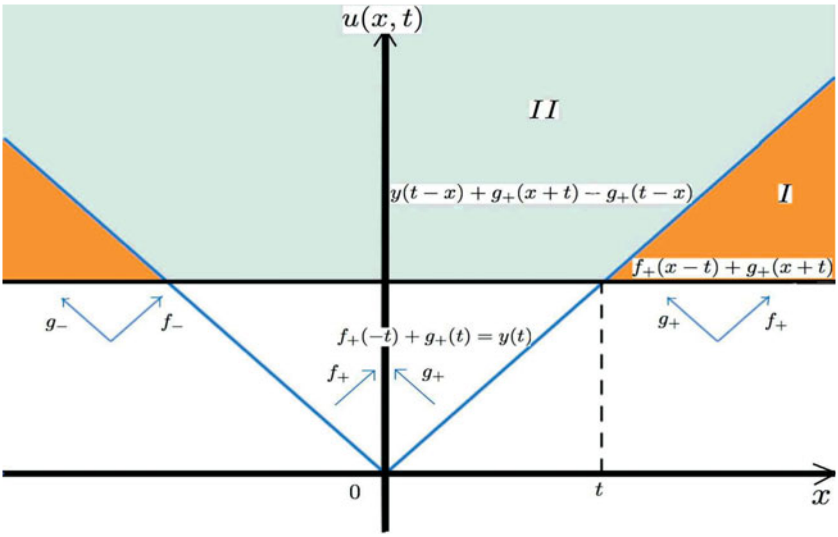

FIGURE 2. Solution to the Lamb system: region I, $x \ge t$ and region II, $x < t$, with $x \ge 0$ and $t > 0$.

Remark 3.3. Similarly to the "direct" reduced equation (3.4), the **"inverse"** one holds,

$$m\ddot{y}(t) = F(y(t)) + 2\dot{y}(t) - 2\dot{w}_{\mathsf{out}}(t), \quad t < 0, \tag{3.8}$$

where the function $w_{\mathsf{out}}(t) := g_-(t) + f_+(-t)$, for $t < 0$ is the *"outgoing wave"*. In this case we have a representation of type (3.7) which contains only the reflected waves g_-, f_+:

$$u(x,t) = \begin{cases} y(x+t) + f_+(x-t) - f_+(-x-t), & 0 < x < -t, \\ y(-x+t) - g_-(-x+t) + g_-(x+t), & t < x < 0, \end{cases} \Bigg| \; t < 0. \tag{3.9}$$

4. Scattering in the Lamb system

The stationary states $S = (s(x), 0) \in \mathcal{E}^0$ if $m = 0$ for (2.6), and $\mathsf{S} = (s(x), 0, 0) \in \mathcal{E}$ if $m > 0$ for (2.7) are evidently determined. We define for every $c \in \mathbb{R}^d$ the constant function

$$s_c(x) = c, \quad x \in \mathbb{R}. \tag{4.1}$$

Then the sets $\mathcal{S}^0$ and $\mathcal{S}$ of all stationary states, $S \in \mathcal{E}^0$ if $m = 0$ (or $\mathsf{S} \in \mathcal{E}$ if $m > 0$) are given by

$$\mathcal{S}^0 = \{S_z = (s_z(x), 0) \mid z \in Z\}, \tag{4.2}$$

$$\mathcal{S} = \{\mathsf{S}_z = (\mathsf{s}_z(x), 0, 0) \mid z \in Z\}, \tag{4.3}$$

where $Z := \{z \in \mathbb{R}^d \mid F(z) = 0\}$. We define $\mathcal{Z} = \{(z, 0) \in \mathbb{R}^{2d} \mid z \in Z\}$.

Definition 4.1. The potential $V(u)$ is "non-degenerate", if the set Z is a discrete subset in $\mathbb{R}^d$.

For $d = 1$ this means that

$$F(u) \not\equiv 0 \text{ on every nonempty interval } c_1 < u < c_2. \tag{4.4}$$

4.1. Convergence to the global attractor

The first result means that the set $\mathcal{S}^0$ (or $\mathcal{S}$) is the **minimal global point attractor** of the system (2.1) in the space $\mathcal{E}^0_F$, (or in the space $\mathcal{E}_F$). Let us denote $\mathcal{E}^0_0 = \{(u, v) \in \mathcal{E}^0\}$ if $m = 0$, $\mathcal{E}_0 = \{(u, v, 0) \in \mathcal{E}\}$ if $m > 0$, and

$$\tilde{W}(t)(u, v) := \begin{cases} \big(W(t)(u, v)\big), & \text{if } m = 0, \\ \big(W(t)(u, v), 0\big), & \text{if } m > 0, \end{cases} \tag{4.5}$$

where $W(t)$ is the dynamical group of free wave equation corresponding to $F(u) \equiv 0$.

Theorem 4.2. [6, 7] *Let all assumptions of Proposition* 3.1 (*or Proposition* 3.2) *hold,* $Y_0 \in \mathcal{E}^0$ *if* $m = 0$ (*or* $\mathsf{Y}_0 \in \mathcal{E}$ *if* $m > 0$) *an initial data. Then,*

i) *The corresponding solution* $Y(t) \in C(\mathbb{R}; \mathcal{E}^0)$, *if* $m = 0$, *to the Cauchy problem* (2.6) (*or the corresponding solution* $\mathsf{Y}(t) \in C(\mathbb{R}; \mathcal{E})$, *if* $m > 0$, *to the Cauchy problem* (2.7)) *converges to the set* $\mathcal{S}^0$ (*or* $\mathcal{S}$) *in the* **local energy semi-norm***:*

$$Y(t) \xrightarrow[t\to\pm\infty]{\mathcal{E}_F^0} \mathcal{S}^0, \quad \text{if} \;\; m = 0, \tag{4.6}$$

or

$$\mathsf{Y}(t) \xrightarrow[t\to\pm\infty]{\mathcal{E}_F} \mathcal{S}, \quad \text{if} \;\; m > 0. \tag{4.7}$$

ii) *There exist the limit stationary states* $S_\pm \in \mathcal{S}^0$ (*or* $\mathsf{S}_\pm \in \mathcal{S}$) *depending on the solution* $Y(t)$ (*or* $\mathsf{Y}(t)$) *such that*

$$Y(t) \xrightarrow[t\to\pm\infty]{\mathcal{E}_F^0} S_\pm, \quad \text{if} \;\; m = 0, \tag{4.8}$$

or

$$\mathsf{Y}(t) \xrightarrow[t\to\pm\infty]{\mathcal{E}_F} S_\pm, \quad \text{if} \;\; m > 0. \tag{4.9}$$

Sketch of the proof. (Theorem 4.2.) We consider $t > 0$, the case $t < 0$ its handled similar. The *stabilization* (4.6) (or (4.7)) follows from the representation (3.7) and from the Lemma:

Lemma 4.3 ([6] if $m = 0$ and [7] if $m > 0$). *Let all assumptions of Theorem* 4.2 *hold. Then,*

A. *If m=0:*

i) *For every solution* $y(t)$ *to the equation* (3.4)

$$y(t) \xrightarrow[t\to+\infty]{} Z. \tag{4.10}$$

ii) *Let, additionally,* Z *be a discrete subset in* $\mathbb{R}^d$. *Then there exists a* $z \in Z$ *such that*

$$y(t) \xrightarrow[t\to+\infty]{} z. \tag{4.11}$$

B. *If* $m > 0$*:*

i) *For every solution* $y(t)$ *to the equation* (3.4)

$$(y(t), \dot{y}(t)) \xrightarrow[t\to+\infty]{} \mathcal{Z}. \tag{4.12}$$

ii) *Let, additionally,* Z *be a discrete subset in* $\mathbb{R}^d$. *Then there exists a* $(z, 0) \in \mathcal{Z}$ *such that*

$$(y(t), \dot{y}(t)) \xrightarrow[t\to+\infty]{} (z, 0). \tag{4.13}$$

Finally, the attraction (4.6) (or (4.7)) implies the convergence (4.8) (or (4.9)). □

4.2. Scattering asymptotic

The next result establish the long time asymptotics of the Lamb system for $m \geq 0$. We write conditions introduced in [6] that restrict to the sets Cauchy data $\mathcal{E}^0$ (or $\mathcal{E}$) that allow to prove the character of convergence to the solutions of Lamb systems. Suppose that the following integral and limits exist

$$\overline{v}_0 := \int_{\mathbb{R}} v_0(\chi)d\chi, \quad u_0^- := \lim_{x\to-\infty} u_0(x), \ u_0^+ := \lim_{x\to+\infty} u_0(x). \tag{4.14}$$

Definition 4.4.

i) The symbol $\mathcal{E}^0_\infty$ denotes the space of pairs $(u, v) \in \mathcal{E}^0$ such that the limits (4.14) exist.

ii) The symbol $\mathcal{E}_\infty$ denotes the space of triples $(u, v, p) \in \mathcal{E}$ such that the limits (4.14) exist.

Theorem 4.5. [6, 7] *Let all assumptions of Proposition* 3.1 (*or Proposition* 3.2) *hold, and* $Y_0 \in \mathcal{E}^0_\infty$ *if* $m = 0$ (*or* $\mathsf{Y}_0 \in \mathcal{E}_\infty$ *if* $m > 0$) *an initial data. We suppose that* Z *is a discrete set in* $\mathbb{R}^d$. *Then*

i) **Scattering asymptotic** *holds:*

$$Y(t) = S_+ + \tilde{W}(t)\Psi_+ + r_+(t), \quad t \to +\infty, \quad \text{if } \ m = 0, \tag{4.15}$$

for some stationary states $S_+ \in \mathcal{S}^0$, *scattering states* $\Psi_+ \in \mathcal{E}^0_0$ *and the remainder is small in the* **global energy norm***:*

$$||r_+(t)||_{\mathcal{E}^0} \xrightarrow[t\to+\infty]{} 0, \tag{4.16}$$

and

$$\mathsf{Y}(t) = \mathsf{S}_+ + \tilde{W}(t)\Psi_+ + r_+(t), \quad t \to +\infty, \quad \text{if } \ m > 0, \tag{4.17}$$

for some stationary states $\mathsf{S}_+ \in \mathcal{S}$, *scattering states* $\Psi_+ \in \mathcal{E}_0$ *and the remainder is small in the* **global energy norm***:*

$$||r_+(t)||_{\mathcal{E}} \xrightarrow[t\to+\infty]{} 0. \tag{4.18}$$

ii) *The outgoing wave* $\tilde{W}(t)\Psi_+$ *converges to zero in the local energy semi-norms:*

$$||\tilde{W}(t)\Psi_+||_{\mathcal{E}^0_F} \xrightarrow[t\to+\infty]{} 0, \quad \Psi_+ \in \mathcal{E}^0_0, \quad \text{if } \ m = 0, \tag{4.19}$$

and

$$||\tilde{W}(t)\Psi_+||_{\mathcal{E}_F} \xrightarrow[t\to+\infty]{} 0, \quad \Psi_+ \in \mathcal{E}_0, \quad \text{if } \ m > 0. \tag{4.20}$$

iii) $\tilde{W}(t)\Psi_+$ *admits the representation:*

$$\tilde{W}(t)\Psi_+ = \begin{cases} \big(\mathsf{w}_{\mathsf{out}}(x,t), \dot{\mathsf{w}}_{\mathsf{out}}(x,t)\big), & \Psi_+ \in \mathcal{E}^0_0, \quad \text{if } \ m = 0, \\ \big(\mathsf{w}_{\mathsf{out}}(x,t), \dot{\mathsf{w}}_{\mathsf{out}}(x,t), 0\big), & \Psi_+ \in \mathcal{E}_0, \quad \text{if } \ m > 0, \end{cases} \tag{4.21}$$

where

$$\mathsf{w}_{\mathsf{out}}(x,t) = C_0 + f_+(x-t) + g_-(x+t), \quad C_0 := \frac{u_0^+ + u_0^- + \overline{v}_0}{2} - 2z_+, \quad z_+ \in Z. \tag{4.22}$$

Remark 4.6.

i) Similar asymptotic hold for $t \to -\infty$. Hence it suffices to prove Theorem 4.5 for $t \to +\infty$ since the Lamb system (2.1) is time reversible.

ii) The "weak" convergence (4.8) and (3.1), (3.2) imply that

$$\mathcal{H}(S_\pm) \leq \mathcal{H}(Y(t)) \equiv \mathcal{H}(Y_0), \;\; t \in \mathbb{R},$$

by the Fatou theorem.

iii) Proposition 3.1, 3.2 and Theorem 4.2 are proved in [4] for one-dimensional oscillator with $d = 1$, $m \geq 0$ and $u_0'(x) = v_0(x) = 0$, $|x| > \text{const}$. In [6] and [7] its consider the arbitrary initial conditions for $m = 0$, $m > 0$ and $d \geq 1$ respectively.

5. Expression of the asymptotic states

Corollary 5.1. *For* $(u_0, v_0) \in \mathcal{E}_\infty^0$ *if* $m = 0$ *(or for* $(u_0, v_0, p_0) \in \mathcal{E}_\infty$ *if* $m > 0$*) the* **asymptotic state:**

$$\Psi_+ = \begin{cases} (\Psi_0, \Pi_0), & \text{if } \; m = 0, \\ (\Psi_0, \Pi_0, 0), & \text{if } \; m > 0, \end{cases} \tag{5.1}$$

are expressed by the formulas:

$$\Psi_0(x) = C_0 + \begin{cases} y(x) + \dfrac{u_0(x) - u_0(-x)}{2} - \dfrac{1}{2}\displaystyle\int_{-x}^{x} v_0(\chi)d\chi, & x \geq 0, \\[2ex] y(-x) + \dfrac{u_0(x) - u_0(-x)}{2} + \dfrac{1}{2}\displaystyle\int_{-x}^{x} v_0(\chi)d\chi, & x \leq 0, \end{cases} \tag{5.2}$$

$$\Pi_0(x) = \begin{cases} y'(x) - \dfrac{u_0'(x) - u_0'(-x)}{2} + \dfrac{v_0(x) - v_0(-x)}{2}, & x \geq 0, \\[2ex] y'(-x) + \dfrac{u_0'(x) - u_0'(-x)}{2} + \dfrac{v_0(x) - v_0(-x)}{2}, & x \leq 0, \end{cases} \tag{5.3}$$

where C_0 *is given by the second formula of* (4.22).

Remark 5.2. Representations (4.22) and (3.8) imply that

$$\mathsf{w}_{\mathsf{out}}(0, t) = C_0 + w_{\mathsf{out}}(t), \quad t > 0.$$

Hence

$$\dot{\mathsf{w}}_{\mathsf{out}}(0, t) = \dot{w}_{\mathsf{out}}(t), \quad t > 0. \tag{5.4}$$

The outgoing wave $\mathsf{w}_{\mathsf{out}}$ admits the D'Alembert representation

$$\mathsf{w}_{\mathsf{out}}(x, t) = \tilde{W}(t)(\Psi_0, \Pi_0) = \frac{\Psi_0(x - t) + \Psi_0(x + t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \Pi_0(\chi)d\chi, \;\; x, t \in \mathbb{R},$$

because $\mathsf{w}_{\mathsf{out}}$ is a solution to the D'Alembert equation.

Definition 5.3. **A.** If $m = 0$:

Let $Y(t) = (u(x,t), v(x,t)) \in C(\mathbb{R}; \mathcal{E}^0_\infty)$ be a solution to (2.6) with $Y(0) = Y_0 \in \mathcal{E}^0_\infty$ be such that the asymptotic (4.15) holds with $S_+(x) = (s_+(x), 0)$, where $s_+(x) \equiv z_+ \in Z$ and $\Psi_+ \in \mathcal{E}^0_0$. Let us set

$$W_+ Y_0 = (\Psi_+, z_+) \in \mathcal{E}^0_0 \times Z. \tag{5.5}$$

The map $W_+ : \mathcal{E}^0_\infty \to \mathcal{E}^0_0 \times Z$ is called a **wave operator**, and (Ψ_+, z_+) the **scattering data**, corresponding to Y_0.

B. If $m > 0$:

Let $\mathsf{Y}(t) = (u(x,t), v(x,t), \dot{y}(t)) \in C(\mathbb{R}; \mathcal{E}_\infty)$ be a solution to (2.7) with $\mathsf{Y}(0) = \mathsf{Y}_0 \in \mathcal{E}_\infty$ be such that the asymptotic (4.17) holds with $\mathsf{S}_+(x) = (\mathsf{s}_+(x), 0, 0)$, where $\mathsf{s}_+(x) \equiv z_+ \in Z$ and $\Psi_+ \in \mathcal{E}_0$. Let us set

$$W_+ \mathsf{Y}_0 = (\Psi_+, z_+) \in \mathcal{E}_0 \times Z. \tag{5.6}$$

The map $W_+ : \mathcal{E}_\infty \to \mathcal{E}_0 \times Z$ is called a **wave operator**, and (Ψ_+, z_+) the **scattering data**, corresponding to Y_0.

5.1. Necessary conditions for the existence of the asymptotic states

We start with description on $\mathrm{Im} W_+$. We describe some properties of the asymptotic states Ψ_0, Π_0 previously constructed (see equations (5.2), (5.3)). Let $Y(t) = \big(u(x,t), v(x,t)\big) \in C(\mathbb{R}; \mathcal{E}^0_\infty)$, be a solution of the Cauchy problem (2.6) with the initial data $Y_0 = \big(u_0(x), v_0(x)\big)$ and let $\mathsf{Y}(t) = \big(u(x,t), v(x,t), \dot{y}(t)\big) \in C(\mathbb{R}; \mathcal{E})$ be a solution of the Cauchy problem (2.7) with $\mathsf{Y}_0 = \big(u_0(x), v_0(x), p_0\big)$.

Proposition 5.4 ([6]).

A. *If $m = 0$: Let $Y_0 \in \mathcal{E}^0_\infty$ and $W_+(Y_0) = (\Psi_+, z_+)$, $z_+ = \lim_{t \to +\infty} y(t)$. Then,*

i) $\Psi_+ \in \mathcal{E}^0_\infty$, *i.e., there exist the finite limits and integral*

$$\Psi_0^- = \lim_{x \to -\infty} \Psi_0(x), \quad \Psi_0^+ = \lim_{x \to +\infty} \Psi_0(x), \quad \overline{\Pi_0} = \int_{-\infty}^{\infty} \Pi_0(\chi) d\chi. \tag{5.7}$$

ii) *The following identity holds:*

$$\Psi_0^+ + \Psi_0^- + \overline{\Pi_0} = 0. \tag{5.8}$$

B. *If $m > 0$: Let $\mathsf{Y}_0 \in \mathcal{E}_\infty$ and $W_+(\mathsf{Y}_0) = (\Psi_+, z_+)$, $z_+ = \lim_{t \to +\infty} y(t)$. Then,*

i) $\Psi_+ \in \mathcal{E}_\infty$, *i.e., there exist the finite limits and integral* (5.7).

ii) *The identity* (5.8) *holds.*

Remark 5.5. Relation (5.8), in both cases ($m = 0$ or $m > 0$), means that the values of Ψ_0^+, Ψ_0^- and $\overline{\Pi_0}$ are not independent.

Let us denote

$$\mathrm{E}^+_\infty := \big\{ \Psi^+ \in \mathcal{E}^0_\infty \ \text{ if } m = 0 \,\big|\, (5.7)\text{–}(5.8) \text{ hold} \big\}.$$

and

$$\mathsf{E}^+_\infty := \big\{ \Psi^+ \in \mathcal{E}_\infty \ \text{ if } m > 0 \,\big|\, (5.7)\text{–}(5.8) \text{ hold} \big\}.$$

Then Proposition 5.4 implies

Corollary 5.6.

$$\mathrm{Im}\; W_+(\mathcal{E}^0_\infty) \subset \mathrm{E}^+_\infty \times Z, \quad \textit{if} \;\; m=0$$

and

$$\mathrm{Im}\; W_+(\mathcal{E}_\infty) \subset \mathsf{E}^+_\infty \times Z, \quad \textit{if} \;\; m>0\ .$$

6. Inverse scattering problem in the Lamb system

In this section we study the inverse problem, namely, we have Ψ_+ and we want to construct the dynamics $Y(t)$ of (2.6) (or $\mathsf{Y}(t)$ of (2.7)) such that the asymptotic (4.15) (or (4.17)) holds. We start from with the reconstruct of Y_0 (or Y_0) via Ψ_+ and $y(t)$.

6.1. Reconstruction of the initial data (surjection of the operator W_+^{-1})

For $\Psi_+ = (\Psi_0, \Pi_0) \in \mathcal{E}^+_\infty$ let us introduce the function

$$S(t) := \mathsf{w}_{\mathsf{out}}(0,t) = \frac{\Psi_0(t)+\Psi_0(-t)}{2} + \frac{1}{2}\int_{-t}^{t} \Pi_0(\chi)d\chi, \quad t\in\mathbb{R}. \tag{6.1}$$

Then

$$\dot S(t) = \dot{\mathsf{w}}_{\mathsf{out}}(t) \in L^2(\mathbb{R},\mathbb{R}^d), \qquad S^+ := \lim_{t\to+\infty} S(t) = 0, \tag{6.2}$$

by (5.4), (3.8) and (5.8).

Theorem 6.1. [8] *Let $Y(t) \in C(\mathbb{R};\mathcal{E}^0_\infty)$ if $m=0$ (or $\mathsf{Y}(t)\in C(\mathbb{R};\mathcal{E}_\infty)$ if $m>0$) be a solution of (2.6) (or (2.7)) with $Y(0)=Y_0\in\mathcal{E}^0_\infty$ if $m=0$ (or $\mathsf{Y}(t)\in\mathcal{E}_\infty$ if $m>0$), and (5.5) (or (5.6)) hold. Then*

i) *The initial conditions are expressed in Ψ_+ and $y(t)=u(0,t)$ by*

$$\begin{aligned} u_0(x) &= \Psi_0(x) + \begin{cases} y(x)-S(x), & x\ge 0,\\ y(-x)-S(-x), & x\le 0,\end{cases}\\ v_0(x) &= \Pi_0(x) + \begin{cases} y'(x)-S'(x), & x\ge 0,\\ y'(-x)-S'(-x), & x\le 0.\end{cases}\end{aligned} \tag{6.3}$$

ii) *The function $y(t)$ satisfies the following conditions*

$$\begin{cases} 0 = F(y(t)) + 2\dot y(t) - 2\dot S(t), & t\ge 0,\\ \dot y(t)\in L^2(\mathbb{R}_+,\mathbb{R}^d),\; y(t) \xrightarrow[t\to+\infty]{} z_+, \end{cases} \quad \textit{if}\; m=0, \tag{6.4}$$

and

$$\begin{cases} m\ddot y(t) = F(y(t)) + 2\dot y(t) - 2\dot S(t), & t\ge 0,\\ \dot y(t)\in L^2(\mathbb{R}_+,\mathbb{R}^d),\; y(t) \xrightarrow[t\to+\infty]{} z_+, \end{cases} \quad \textit{if}\; m>0. \tag{6.5}$$

Remark 6.2. For any given $\Psi_+ = (\Psi_0, \Pi_0) \in \mathrm{E}^+_\infty$ (or $\Psi_+ = (\Psi_0, \Pi_0, 0) \in \mathsf{E}^+_\infty$) and $y(t) \in C(\overline{\mathbb{R}_+}; \mathbb{R}^d)$, the formulas (6.3) imply (5.2) and (5.3) with $C_0 = \Psi_0(0) - y(0)$.

The next result establishes that the conditions (5.7) are sufficient for the existence of dynamics in the Lamb's problem with scattering asymptotic (4.15) (or (4.17)) provided that the "inverse" reduced differential equation (6.4) has an appropriate solution.

6.2. Characterization on the asymptotic states

Conjecture 6.3. *Let* $(\Psi_+, z_+) \in \mathrm{E}^+_\infty \times Z$ (*or* $(\Psi_+, z_+) \in \mathsf{E}^+_\infty \times Z$) *and the following condition hold:* **there exists a trajectory** $y(t)$ **satisfying** (6.4) **(or** 6.5**), with** $S(t)$ **given by** (6.1)**.** *Then there exists* $Y_0 \in \mathcal{E}^0_\infty$ (*or* $\mathsf{Y}_0 \in \mathcal{E}_\infty$) *such that* (5.5) (*or* (5.6)) *hold.*

Remark 6.4. We choose the inverse reduced equation (6.4) for the characterization of the asymptotic states since the term $\dot{S}(t)$ is expressed in the scattering data Ψ_+ by (6.1).

The following results establishes the asymptotic completeness for some particular cases:

6.2.1. The case $m = 0$ and finite scattering data.

Theorem 6.5. [8] *Let the function* $\Psi_+(x) = \big(\Psi_0(x), \Pi_0(x)\big) \in \mathrm{E}^+_\infty$ *has a compact support,* $d \geq 1$, *and the force field* F *satisfy conditions* (3.1). *Then for arbitrary* $z_+ \in Z$ *there exist* $Y_0 \in \mathcal{E}^0_\infty$ *such that* (5.5) *hold.*

6.2.2. The case $m = 0$ and arbitrary scattering data: one-dimensional oscillator. We consider case $d = 1$. Suppose that $z_+ \in Z$ is nondegenerate stationary state with

$$F'(z_+) \neq 0. \tag{6.6}$$

Theorem 6.6. [8] *Let* $d = 1$, $\Psi_+(x) = \big(\Psi_0(x), \Pi_0(x)\big) \in \mathrm{E}^+_\infty$ *and* (6.6). *Then there exists* $Y_0 \in \mathcal{E}_\infty$ *such that* (5.5) *hold.*

7. Example: $F(u) = -u$

In this section we will prove that the "inverse" reduced equation (6.4) in the case $m > 0$ admits a continuous solution $y(t)$, $t \geq 0$ with $\dot{y}(t) \in L^2(\mathbb{R}_+; \mathbb{R}^d)$ for the force field $F(u) = -u$. Note that this function satisfies the conditions (3.1).

Let us consider the following Cauchy problem for the operator $\mathcal{L}\Big(\frac{d}{dt}\Big) := m\frac{d^2}{dt^2} - 2\frac{d}{dt} + 1$, $m > 0$:

$$\begin{cases} \mathcal{L}[y](t) := \mathcal{L}\Big(\dfrac{d}{dt}\Big)[y(t)] = f(t), \quad t \geq 0, \quad f \in L^2(\mathbb{R}_+, \mathbb{R}^d) \\ \quad y(0) = a, \quad \dot{y}(0) = b, \quad a, b \in \mathbb{R}. \end{cases} \tag{7.1}$$

We seek solutions of this problem from the class $C(\overline{\mathbb{R}_+};\mathbb{R}^d)$ with $\dot{y}\in L^2(\mathbb{R}_+;\mathbb{R}^d)$. For $y(t)\in L^1_{\mathrm{loc}}(\overline{\mathbb{R}_+};\mathbb{R}^d)$ we define the distribution $y_0(t)\in\mathcal{D}'(\mathbb{R};\mathbb{R}^d)$ by the formula

$$y_0(t)=\begin{cases} y(t), & x\in\overline{\mathbb{R}_+},\\ 0, & x\notin\overline{\mathbb{R}_+}.\end{cases} \tag{7.2}$$

Lemma 7.1 (Cf. [10]). *Let* $u\in C(\overline{\mathbb{R}_+};\mathbb{R}^d)$ *with* $u''\in L^1_{\mathrm{loc}}(\overline{\mathbb{R}_+};\mathbb{R}^d)$ *we have the following representations in the sense of* $\mathcal{D}'(\mathbb{R};\mathbb{R}^d)$

$$u'(x)=u(0)\delta(x)+[u'(x)],\quad x\in\mathbb{R}, \tag{7.3}$$

$$u''(x)=u(0)\delta'(x)+u'(0)\delta(x)+[u''(x)],\quad x\in\mathbb{R}, \tag{7.4}$$

where $[u'(x)]$ *and* $[u''(x)]$ *are the usual derivatives in* $\mathbb{R}_+$.

Using (7.3) and (7.4) we get

$$\mathcal{L}[y_0](t)=am\dot{\delta}(t)+(bm-2a)\delta(t)+f_0(t),\quad t\in\mathbb{R}. \tag{7.5}$$

Denote by $\mathcal{S}(\overline{\mathbb{R}_+};\mathbb{R}^d)$ the subspace of the Schwartz space of rapidly decreasing functions with supports belonging to $\overline{\mathbb{R}_+}$. We use the complex Fourier-Laplace transform $\mathcal{F}:\mathcal{S}'(\overline{\mathbb{R}_+};\mathbb{R}^d)\to\mathcal{S}'(\overline{\mathbb{R}_+};\mathbb{R}^d)$, defined by

$$\tilde{g}(z)\equiv\mathcal{F}_{t\to z}[g](z):=\int_{\mathbb{R}_+}\mathrm{e}^{itz}g(t)dt,\quad z\in\mathbb{C}_+, \tag{7.6}$$

for $g\in\mathcal{S}(\overline{\mathbb{R}_+};\mathbb{R}^d)$ and extended by continuity to $\mathcal{S}'(\overline{\mathbb{R}_+};\mathbb{R}^d)$. Applying this transform to the identity (7.5), we obtain

$$\big(-mz^2+2iz+1\big)\tilde{y}_0(z)=-iamz+(bm-2a)+\tilde{f}_0(z),\quad z\in\mathbb{C}_+. \tag{7.7}$$

Hence

$$\tilde{y}_0(z)=\frac{-iamz+(bm-2a)+\tilde{f}_0(z)}{-mz^2+2iz+1},\quad z\in\mathbb{C}_+, \tag{7.8}$$

is analytic in $\mathbb{C}_+$ if a and b satisfy the system

$$\begin{cases} a(-imz_1-2)+bm=-\tilde{f}_0(z_1),\\ a(-imz_2-2)+bm=-\tilde{f}_0(z_2),\end{cases}$$

with z_1 and z_2 are the roots of the equation $-mz^2+2iz+1$, i.e.,

$$\begin{cases} a=\dfrac{\tilde{f}_0(z_1)-\tilde{f}_0(z_2)}{im(z_1-z_2)}\,,\\[2ex] b=\dfrac{(imz_2+2)\tilde{f}_0(z_1)-(imz_2+2)\tilde{f}_0(z_2)}{im^2(z_1-z_2)}\,.\end{cases}$$

Substituting the obtained values of a and b to (7.8) and using the Paley-Wiener Theorem [10], we obtain in the case $m\neq\frac{1}{4}$

$$y_0(t)=\mathcal{F}^{-1}_{z\to t}\Big\{\frac{\tilde{f}_0(z_1)(z-z_2)-\tilde{f}_0(z_2)(z-z_1)+(z_2-z_1)\tilde{f}_0(z)}{-m(z-z_1)(z-z_2)(z_2-z_1)}\Big\},\quad t\in\mathbb{R}. \tag{7.9}$$

Thus, the solution to the Cauchy problem (7.1) is

$$y(t) = y_0(t)|_{\mathbb{R}_+}. \tag{7.10}$$

Using (7.8) we have

$$(\dot{y}(t))\check{}(z) = \frac{i}{m(z_2 - z_1)}\left[\frac{z_1\tilde{f}_0(z_1)}{z - z_1} - \frac{z_2\tilde{f}_0(z_2)}{z - z_2} + \frac{(z_2 - z_1)z\tilde{f}_0(z)}{(z - z_1)(z - z_2)}\right], \quad z \in \mathbb{C}_+. \tag{7.11}$$

Suppose that $f \in L^2(\mathbb{R};\mathbb{R}^d)$. Then $\tilde{h}(\tau + i\kappa) = (\dot{y}(t))\check{}\,(\tau + i\kappa) \in L^2(\mathbb{R}_\tau;\mathbb{R}^d)$ for each $\kappa \geq 0$ and

$$||\tilde{h}(\tau + i\kappa)||_{L^2(\mathbb{R}_\tau,\mathbb{R}^d)} \leq C, \quad \kappa \geq 0.$$

By Paley-Wiener Theorem is implies that $\tilde{h}(x) \in L^2(\mathbb{R}_+;\mathbb{R}^d)$. Therefore, the Cauchy problem (7.1) admits a continuous solution with the derivative from $L^2(\mathbb{R}_+;\mathbb{R}^d)$. Taking $f = \dot{S}$ we obtain that the equation (6.4) admits a solution with the properties in Theorem 6.3. Hence the solution the inverse scattering problem of Lamb system in this case is solvable.

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Toeplitz Algebras in Quantum Hopf Fibrations

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Abstract. The paper presents applications of Toeplitz algebras in Noncommutative Geometry. As an example, a quantum Hopf fibration is given by gluing trivial U(1) bundles over quantum discs (or, synonymously, Toeplitz algebras) along their boundaries. The construction yields associated quantum line bundles over the generic Podleś spheres which are isomorphic to those from the well-known Hopf fibration of quantum SU(2). The relation between these two versions of quantum Hopf fibrations is made precise by giving an isomorphism in the category of right U(1)-comodules and left modules over the C*-algebra of the generic Podleś spheres. It is argued that the gluing construction yields a significant simplification of index computations by obtaining elementary projections as representatives of K-theory classes.

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1. Introduction

In Noncommutative Geometry, the Toeplitz algebra has a fruitful interpretation as the algebra of continuous function on the quantum disc [10]. In this picture, the description of the Toeplitz algebra as the C*-algebra extension of continuous functions on the circle by the compact operators corresponds to an embedding of the circle into the quantum disc. Analogous to the classical case, one can construct "topologically" non-trivial quantum spaces by taking trivial fibre bundles over two quantum discs and gluing them along their boundaries. Here, the gluing procedure is described by a fibre product in an appropriate category (C*-algebras, finitely generated projective modules, etc.). This approach has been applied successfully to the construction of line bundles over quantum 2-spheres [2, 8, 17] and to the description of quantum Hopf fibrations [1, 3, 7, 9]. One of the advantages of the fibre product approach is that it provides an effective tool for simplifying index

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computations. This has been discussed in [17] on the example of the Hopf fibration of quantum SU(2) over the generic Podleś spheres [14]. Whilst earlier index computations for quantum 2-spheres relied heavily on the index theorem [4, 6], the fibre product approach in [17] allowed to compute the index pairing directly by producing simpler representatives of K-theory classes.

The description of quantum line bundles in [17] bears a striking analogy to the classical case: the same transition functions are used to glue the trivial bundles over the (quantum) disc along their boundaries. However, the link between the fibre product approach of quantum line bundles and the Hopf fibration of quantum SU(2) has been established only at a "K-theoretic level", i.e., it has been shown that the corresponding projective modules are Murray-von Neumann equivalent. The present work will give a more geometrical picture of the quantum Hopf fibration. Analogous to the classical case, we will construct a non-trivial U(1) quantum principal bundle over the generic Podleś spheres such that the associated line bundles are the previously obtained quantum line bundles. Here, a quantum principal bundle is described by a Hopf-Galois extension (see the preliminaries). It turns out that our U(1) quantum principal bundle is isomorphic to a quantum 3-sphere from [3]. As an application of the fibre product approach, we will show that the associated quantum line bundles are isomorphic to projective modules given by completely elementary one-dimensional projections which leads to a significant simplification of index computations.

It is known that the Hopf fibration of quantum SU(2) over the generic Podleś spheres is not given by a Hopf-Galois extension but only by a so-called coalgebra Galois extension (that is, U(1) is only considered as a coalgebra). In the present paper, we will establish a relation between both versions of a quantum Hopf fibration by describing an explicit isomorphism in the category of right U(1)-comodules and left modules over the C*-algebra of the generic Podleś spheres. Clearly, this isomorphism cannot be turned into an algebra isomorphism of quantum 3-spheres since otherwise the Hopf fibration of quantum SU(2) over the generic Podleś spheres would be a Hopf-Galois extension.

2. Preliminaries

2.1. Coalgebras and Hopf algebras

A coalgebra is a vector space C over a field $\mathbb{K}$ equipped with two linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : S \to \mathbb{K}$, called the comultiplication and the counit, respectively, such that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{2.1}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta. \tag{2.2}$$

A (right) corepresentation of a coalgebra C on a $\mathbb{K}$-vector space V is a linear mapping $\Delta_V : V \to V \otimes C$ satisfying

$$(\Delta_V \otimes \mathrm{id}) \circ \Delta_V = (\mathrm{id} \otimes \Delta) \circ \Delta_V, \quad (\mathrm{id} \otimes \varepsilon) \circ \Delta_V = \mathrm{id}. \tag{2.3}$$

We then refer to V as a right C-comodule. The corepresentation is said to be irreducible if $\{0\}$ and V are the only invariant subspaces. A linear mapping ϕ between right C-comodules V and W is called colinear, if $\Delta_W \circ \phi = (\phi \otimes \mathrm{id}) \circ \Delta_V$.

A Hopf algebra A is a unital algebra and coalgebra such that Δ and ε are algebra homomorphism, together with a linear mapping $\kappa : A \to A$, called the antipode, such that

$$m \circ (\kappa \otimes \mathrm{id}) \circ \Delta(a) = \varepsilon(a) = m \circ (\mathrm{id} \otimes \kappa) \circ \Delta(a), \quad a \in A, \tag{2.4}$$

where $m : A \otimes A \to A$ denotes the multiplication map.

We say that C and A are a *-coalgebra and a *-Hopf algebra, respectively, if C and A carry an involution such that Δ becomes a *-morphism. This immediately implies that $\varepsilon(x^*) = \overline{\varepsilon(x)}$. A finite-dimensional corepresentation $\Delta_V : V \to V \otimes A$ is called unitary, if there exists a linear basis $\{e_1, \ldots, e_n\}$ of V such that $\Delta_V(e_i) = \sum_{j=1}^n e_j \otimes v_{ji}$ and $\sum_{j=1}^n v_{jk}^* v_{ji} = \delta_{ki}$, where δ_{kj} denotes the Kronecker symbol. The elements v_{ij} are called matrix coefficients. A Hopf *-algebra A is called a compact quantum group algebra if it is the linear span of all matrix coefficients of irreducible finite-dimensional unitary corepresentations. It can be shown that then A admits a C*-algebra completion H in the universal C*-norm (that is, the supremum of the norms of all bounded irreducible Hilbert space *-representations). We call H also a compact quantum group and refer to the dense subalgebra A as its Peter-Weyl algebra. The counit of A has then a unique extension to H, and Δ has a unique extension to a *-homomorphism $\Delta : H \to H \bar{\otimes} H$, where $H \bar{\otimes} H$ denotes the least C*-completion of the algebraic tensor product.

The main example in this paper will be $H = \mathcal{C}(\mathrm{S}^1)$, the C*-algebra of continuous functions on the unit circle S^1. It is a compact quantum group with comultiplication $\Delta(f)(p, q) = f(pq)$, counit $\varepsilon(f) = f(1)$ and antipode $\kappa(f)(p) = f(p^{-1})$. Note that Δ, ε and κ are given by pullbacks of the group operations of $\mathrm{S}^1 = \mathrm{U}(1)$. Let $U \in \mathcal{C}(\mathrm{S}^1)$, $U(\mathrm{e}^{\mathrm{i}\phi}) = \mathrm{e}^{\mathrm{i}\phi}$, denote the unitary generator of $\mathcal{C}(\mathrm{S}^1)$. Then the Peter-Weyl algebra of H is given by $\mathcal{O}(\mathrm{U}(1)) = \mathrm{span}\{U^N : N \in \mathbb{Z}\}$ with $\Delta(U^N) = U^N \otimes U^N$, $\varepsilon(U^N) = 1$ and $\kappa(U^N) = U^{-1}$. Note also that the irreducible unitary corepresentations of $\mathcal{O}(\mathrm{U}(1))$ are all one-dimensional and are given by $\Delta_{\mathbb{C}}(1) = 1 \otimes U^N$.

From the previous paragraph, it becomes clear why noncommutative compact quantum groups are regarded as generalizations of function algebras on compact groups. We give now the definition for a quantum analogue of principal bundles. First we remark that a group action on a topological space corresponds to a coaction of a quantum group or, more generally, to a coaction of a coalgebra. Now let A be a Hopf algebra, P a unital algebra, and $\Delta_P : P \to P \otimes A$ a corepresentation which is also an algebra homomorphism (one says that P is a right A-comodule algebra). Then the space of coinvariants

$$P^{\mathrm{co}A} := \{b \in P : \Delta(b) = b \otimes 1\}$$

is an algebra considered as a function algebra on the base space, and P plays the role of a function algebra on the total space. If A is a Hopf *-algebra and P is a *-algebra, we require Δ to be a *-homomorphism so that B becomes a unital *-subalgebra of P.

If $\Delta : P \to P \otimes C$ is a corepresentation of a coalgebra C, then we set

$$P^{\mathrm{co}C} := \{\, b \in P : \Delta(bp) = b\Delta(p) \ \text{ for all } \ p \in P \,\}$$

with multiplication $b(p \otimes c) = bp \otimes c$ on the left tensor factor. Again, $P^{\mathrm{co}C}$ is a subalgebra of P. In our examples, there will be a group like element $e \in C$ (that is, $\Delta(e) = e \otimes e$) such that $\Delta(1) = 1 \otimes e$ and

$$P^{\mathrm{co}C} = B := \{\, b \in P : \Delta(b) = b \otimes e \,\}.$$

If P and C carry an involution, Δ_P is a *-morphism and $e^* = e$, then B is a *-subalgebra of P.

Analogous to right corepresentations, one defines left corepresentations ${}_V\Delta : V \to C \otimes V$. The associated (quantum) vector bundles are given by the cotensor product $P \,\square_C\, V$, where

$$P \,\square_C\, V := \{\, x \in P \otimes V : (\Delta_P \otimes \mathrm{id})(x) = (\mathrm{id} \otimes {}_V\Delta)(x) \,\}.$$

Obviously, $P \,\square_C\, V$ is a left $P^{\mathrm{co}C}$-module. For the one-dimensional representation ${}_{\mathbb{C}}\Delta(1) = U^N \otimes 1$, this module is equivalent to

$$P_N := \{\, p \in P : \Delta_P(p) = p \otimes U^N \,\}$$

and is considered as a (quantum) line bundle.

2.2. Pullback diagrams and fibre products

The purpose of this section is to collect some elementary facts about fibre products. For simplicity, we start by considering the category of vector spaces. Let $\pi_0 : A_0 \to A_{01}$ and $\pi_1 : A_1 \to A_{01}$ be vector spaces morphisms. Then the fibre product $A := A_0 \times_{(\pi_0,\pi_1)} A_1$ is defined by the pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \mathrm{pr}_1\ } & A_1 \\ {\scriptstyle \mathrm{pr}_0}\big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\ A_0 & \xrightarrow{\ \pi_0\ } & A_{01}\ . \end{array} \tag{2.5}$$

Up to a unique isomorphism, A is given by

$$A = \{(a_0, a_1) \in A_0 \times A_1 : \pi_0(a_0) = \pi_1(a_1)\}\,, \tag{2.6}$$

where the morphisms $\mathrm{pr}_0 : A \to A_0$ and $\mathrm{pr}_1 : A \to A_1$ are the left and right projections, respectively. In this paper, we will consider fibre products in the following categories:

- If $\pi_0 : A_0 \to A_{01}$ and $\pi_1 : A_1 \to A_{01}$ are morphisms of *-algebras, then the fibre product $A_0 \times_{(\pi_0,\pi_1)} A_1$ is a *-algebra with componentwise multiplication and involution.
- If we consider the pullback diagram (2.5) in the category of unital C^*-algebras, then $A_0 \times_{(\pi_0,\pi_1)} A_1$ will be a unital C^*-algebra.

- If B is an algebra and $\pi_0 : A_0 \to A_{01}$ and $\pi_1 : A_1 \to A_{01}$ are morphisms of left B-modules, then the fibre product $A := A_0 \times_{(\pi_0,\pi_1)} A_1$ is a left B-module with left action $b.(a_0, a_1) = (b.a_0, b.a_1)$, where $b \in B$ and the dot denotes the left action.
- If we consider the pullback diagram (2.5) in the category of right C-comodules (or right H-comodule algebras), then $A := A_0 \times_{(\pi_0,\pi_1)} A_1$ will be a right C-comodule (or a right H-comodule algebra) with the coaction given by $\Delta_A(a_1, a_2) = (\Delta_{A_1}(a_1), 0) + (0, \Delta_{A_2}(a_2))$.

Finally we remark that if B_0, B_1 and B_{01} are dense subalgebras of C^*-algebras A_0, A_1 and A_{01}, respectively, and π_0 and π_1 restrict to morphisms $\pi_0 : B_0 \to B_{01}$ and $\pi_1 : B_1 \to B_{01}$, then $B_0 \times_{(\pi_0,\pi_1)} B_1$ is not necessarily dense in $A_0 \times_{(\pi_0,\pi_1)} A_1$. A useful criterion for this to happen can be found in [9, Theorem 1.1]. It suffices that $\pi_1\restriction_{B_1} : B_1 \to B_{01}$ is surjective and $\ker(\pi_1) \cap B_1$ is dense in $\ker(\pi_1)$.

2.3. Disc-type quantum 2-spheres

From now on we will work over the complex numbers and q will denote a real number from the interval $(0, 1)$.

The *-algebra $\mathcal{O}(\mathrm{D}^2_q)$ of polynomial functions on the quantum disc is generated by two generators z and z^* with relation

$$z^*z - qzz^* = 1 - q. \tag{2.7}$$

A complete list of bounded irreducible *-representations of $\mathcal{O}(\mathrm{D}^2_q)$ can be found in [10]. First, there is a faithful representation on the Hilbert space $\ell^2(\mathbb{N}_0)$. On an orthonormal basis $\{e_n : n \in \mathbb{N}_0\}$, the action of the generators reads as

$$ze_n = \sqrt{1 - q^{n+1}}Se_n, \quad z^*e_n = \sqrt{1 - q^n}S^*e_n, \tag{2.8}$$

where

$$Se_n = e_{n+1},$$

denotes the shift operator on $\ell^2(\mathbb{N}_0)$.

Next, there is a 1-parameter family of irreducible *-representations ρ_u on $\mathbb{C}$, where $u \in \mathrm{S}^1 = \{x \in \mathbb{C} : |x| = 1\}$. They are given by assigning

$$\rho_u(z) = u, \qquad \rho_u(z^*) = \bar{u}.$$

The set of these representations is considered as the boundary S^1 of the quantum disc consisting of "classical points".

The universal C*-algebra of $\mathcal{O}(\mathrm{D}^2_q)$ is well known. It has been discussed by several authors (see, e.g., [10, 12, 16]) that it is isomorphic to the Toeplitz algebra $\mathcal{T}$. Here, it is convenient to view the Toeplitz algebra $\mathcal{T}$ as the universal C*-algebra generated by S and S^* in $\mathrm{B}(\ell^2(\mathbb{N}_0))$. Then above *-representation on $\ell^2(\mathbb{N}_0)$ becomes simply an embedding.

Another characterization is given by the C*-extension

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N}_0)) \longrightarrow \mathcal{T} \overset{\sigma}{\longrightarrow} \mathcal{C}(\mathrm{S}^1) \longrightarrow 0,$$

where $\sigma : \mathcal{T} \to \mathcal{C}(\mathrm{S}^1)$ is the so-called symbol map and corresponds, in the classical case, to an embedding of S^1 into the complex unit disc. Let again $U(\mathrm{e}^{\mathrm{i}\phi}) = \mathrm{e}^{\mathrm{i}\phi}$ denote the unitary generator of $\mathcal{C}(\mathrm{S}^1)$. Then the symbol map is completely determined by setting $\sigma(z) = U$.

We can now construct a quantum 2-sphere $\mathcal{C}(\mathrm{S}^2_q)$ by gluing two quantum discs along their boundaries. The gluing procedure is described by the fibre product $\mathcal{T} \times_{(\sigma,\sigma)} \mathcal{T}$, where $\mathcal{T} \times_{(\sigma,\sigma)} \mathcal{T}$ is defined by the following pullback diagram in the category of C*-algebras:

$$\begin{array}{ccc} \mathcal{T} \underset{(\sigma,\sigma)}{\times} \mathcal{T} & \xrightarrow{\mathrm{pr}_1} & \mathcal{T} \\ {\scriptstyle \mathrm{pr}_0}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \mathcal{T} & \xrightarrow{\sigma} & \mathcal{C}(\mathrm{S}^1). \end{array} \tag{2.9}$$

Up to isomorphism, the C*-algebra $\mathcal{C}(\mathrm{S}^2_q) := \mathcal{T} \times_{(\sigma,\sigma)} \mathcal{T}$ is given by

$$\mathcal{C}(\mathrm{S}^2_q) = \{\, (a_1, a_2) \in \mathcal{T} \times \mathcal{T} : \sigma(a_1) = \sigma(a_2) \,\}. \tag{2.10}$$

In the classical case, complex line bundles with winding number $N \in \mathbb{Z}$ over the 2-sphere can be constructed by taking trivial bundles over the northern and southern hemispheres and gluing them together along the boundary via the map $U^N : \mathrm{S}^1 \to \mathrm{S}^1$, $U^N(\mathrm{e}^{\mathrm{i}\phi}) = \mathrm{e}^{\mathrm{i}\phi N}$. In [17], the same construction has been applied to to the quantum 2-sphere $\mathcal{C}(\mathrm{S}^2_q)$. The roles of the northern and southern hemispheres are played by two copies of the quantum disc, and the transition function along the boundaries remains the same. This construction can be expressed by the following pullback diagram:

$$\begin{array}{ccccc} & & \mathcal{T} \underset{(U^N\sigma,\sigma)}{\times} \mathcal{T} & & \\ & {\scriptstyle \mathrm{pr}_0}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_1} & \\ \mathcal{T} & & & & \mathcal{T} \\ {\scriptstyle \sigma}\downarrow & & & & \downarrow{\scriptstyle \sigma} \\ \mathcal{C}(\mathrm{S}^1) & & \xrightarrow[f \mapsto U^N f]{} & & \mathcal{C}(\mathrm{S}^1). \end{array} \tag{2.11}$$

So, up to isomorphism, we have

$$\mathcal{T} \times_{(U^N\sigma,\sigma)} \mathcal{T} \cong \{\, (a_0, a_1) \in \mathcal{T} \times \mathcal{T} \,:\, U^N\sigma(a_0) = \sigma(a_1) \,\}. \tag{2.12}$$

It follows directly from Equation (2.10) that $\mathcal{T} \times_{(U^N\sigma,\sigma)} \mathcal{T}$ is a $\mathcal{C}(\mathrm{S}^2_q)$-(bi)module. This can also be seen from the general pullback construction by equipping $\mathcal{T}$ and $\mathcal{C}(\mathrm{S}^1)$ with the structure of a left $\mathcal{C}(\mathrm{S}^2_q)$-module. Explicitly, for $(a_0, a_1) \in \mathcal{C}(\mathrm{S}^2_q)$, one defines $(a_0, a_1).a = a_0 a$ for $a \in \mathcal{T}$ on the left side, $(a_0, a_1).a = a_1 a$ for $a \in \mathcal{T}$ on the right side, and $(a_0, a_1).b = \sigma(a_0)b = \sigma(a_1)b$ for $b \in \mathcal{C}(\mathrm{S}^1)$.

To determine the K-theory and K-homology of $\mathcal{C}(\mathrm{S}^2_q)$, we may use the results of [12]. There it is shown that $\mathrm{K}_0(\mathcal{C}(\mathrm{S}^2_q)) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\mathrm{K}^0(\mathcal{C}(\mathrm{S}^2_q)) \cong \mathbb{Z} \oplus \mathbb{Z}$. The two generators of the K_0-group can be chosen to be the class [1] of the unit element of $\mathcal{C}(S^2_q)$, and the class $[(0, 1 - SS^*)]$.

Describing an even Fredholm module by a pair of representations on the same Hilbert space such that the difference is a compact operator, one generator of $K^0(\mathcal{C}(S^2_q))$ is obviously given by the class $[(\mathrm{pr}_1, \mathrm{pr}_0)]$ on the Hilbert space $\ell^2(\mathbb{N}_0)$. A second generator is $[(\pi_+ \circ \sigma\, , \pi_- \circ \sigma)]$, where σ denotes the symbol map and $\pi_\pm : \mathcal{C}(S^1) \to B(\ell^2(\mathbb{Z}))$ is given by

$$\begin{aligned} &\pi_+(U)e_n = e_{n+1}, \quad n \in \mathbb{Z}, \\ &\pi_-(U)e_n = e_{n+1}, \quad n \in \mathbb{Z} \setminus \{-1, 0\}, \quad \pi_-(U)e_{-1} = e_1, \quad \pi_-(U)e_0 = 0 \end{aligned} \tag{2.13}$$

on an orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ of $\ell^2(\mathbb{Z})$. Note that the representation π_- is non-unital: $\pi_-(1)$ is the projection onto $\mathrm{span}\{e_n : n \in \mathbb{Z} \setminus \{0\}\}$.

2.4. Quantum 3-spheres and quantum Hopf fibrations

First we follow [7] and introduce the coordinate ring of a Heegaard-type quantum 3-sphere $\mathcal{O}(S^3_{pq})$, $p, q \in (0, 1)$ as the *-algebra generated by a, a^*, b, b^* subjected to the relations

$$\begin{aligned} &a^*a - qaa^* = 1 - q, \quad b^*b - pbb^* = 1 - p, \\ &(1 - aa^*)(1 - bb^*) = 0, \quad ab = ba, \quad a^*b = ba^*. \end{aligned} \tag{2.14}$$

Its universal C*-algebra (i.e., the closure of $\mathcal{O}(S^3_{pq})$ in the universal C*-norm given by the supremum over all bounded Hilbert space representations) will be denoted by $\mathcal{C}(S^3_{pq})$.

One can easily verify that the coaction $\Delta_{\mathcal{O}(S^3_{pq})} : \mathcal{O}(S^3_{pq}) \to \mathcal{O}(S^3_{pq}) \otimes \mathcal{O}(U(1))$ given by

$$\Delta_{\mathcal{O}(S^3_{pq})}(a) = a \otimes U^*, \quad \Delta_{\mathcal{O}(S^3_{pq})}(b) = b \otimes U$$

turns $\mathcal{O}(S^3_{pq})$ into a $\mathcal{O}(U(1))$-comodule *-algebra. Its *-subalgebra of $\mathcal{O}(U(1))$-coinvariants $\mathcal{O}(S^2_{pq}) := \mathcal{O}(S^3_{pq})^{\mathrm{co}\mathcal{O}(U(1))}$ is generated by

$$A := 1 - aa^*, \quad B := 1 - bb^*, \quad R := ab$$

with involution $A^* = A$, $B^* = B$ and commutation relations

$$R^*R = 1 - qA - pB, \quad RR^* = 1 - A - B, \quad AR = qRA, \quad BR = pRB, \quad AB = 0.$$

Note that $\mathcal{O}(S^2_{pq})$ can also be considered as a *-subalgebra of $\mathcal{C}(S^2_q)$ from (2.10) by setting

$$A = (1 - zz^*, 0), \quad B = (0, 1 - yy^*), \quad R = (z, y),$$

where y and z denote the generators of the quantum discs $\mathcal{O}(D^2_p)$ and $\mathcal{O}(D^2_q)$, respectively, satisfying the defining relation (2.7). Using the fact that $\mathcal{O}(D^2_q)$ is dense in the Toeplitz algebra $\mathcal{T}$ for all $q \in (0, 1)$, and the final remark of Section 2.2, one easily proves that $\mathcal{C}(S^2_q) = \mathcal{T} \times_{(\sigma,\sigma)} \mathcal{T}$ is the universal C*-algebra of $\mathcal{O}(S^2_{pq})$.

For $N \in \mathbb{Z}$, let

$$L_N := \{\, p \in \mathcal{O}(S^3_{pq}) \,:\, \Delta_{\mathcal{O}(S^3_{pq})}(p) = p \otimes U^N \,\} \tag{2.15}$$

denote the associated quantum line bundles. It has been shown in [7] that L_N is isomorphic to $\mathcal{O}(\mathrm{S}^2_{pq})^{|N|+1}E_N$, where

$$E_N = X_N\, Y_N^{\mathrm{t}} \in \mathrm{Mat}_{|N|+1,|N|+1}(\mathcal{O}(\mathrm{S}^2_{pq})) \tag{2.16}$$

and, for $n \in \mathbb{N}$,

$$X_{-n} = (b^{*n}, ab^{*n-1}, \ldots, a^n)^{\mathrm{t}}, \qquad X_n = (a^{*n}, ba^{*n-1}, \ldots, b^n)^{\mathrm{t}},$$

$$Y_{-n} = \Big(\tbinom{n}{0}_p p^n A^n b^n\,,\ \tbinom{n}{1}_p p^{n-1}A^{n-1}b^{n-1}a^*\,,\ \ldots\,,\ \tbinom{n}{n}_p a^{*n}\Big)^{\mathrm{t}},$$

$$Y_n = \Big(\tbinom{n}{0}_q q^n B^n a^n\,,\ \tbinom{n}{1}_q q^{n-1}B^{n-1}a^{n-1}b^*\,,\ \ldots\,,\ \tbinom{n}{n}_q b^{*n}\Big)^{\mathrm{t}},$$

with

$$\tbinom{n}{0}_x = \tbinom{n}{n}_x := 1, \quad \tbinom{n}{k}_x := \tfrac{(1-x)\ldots(1-x^n)}{(1-x)\ldots(1-x^k)(1-x)\ldots(1-x^{n-k})},\ \ 0<k<n,\ \ x \in (0,1).$$

That E_N is indeed an idempotent follows from $Y_N^{\mathrm{t}}\, X_N = 1$ which can be verified by direct computations.

Now we consider a much more prominent example of a quantum Hopf fibration. The *-algebra $\mathcal{O}(\mathrm{SU}_q(2))$ of polynomial functions on the quantum group $\mathrm{SU}_q(2)$ is generated by α, β, γ, δ with relations

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \quad \beta\gamma = \gamma\beta,$$
$$\alpha\delta - q\beta\gamma = 1, \quad \delta\alpha - q^{-1}\beta\gamma = 1,$$

and involution $\alpha^* = \delta$, $\beta^* = -q\gamma$. This is actually a Hopf *-algebra with the Hopf structure Δ, ε, κ. Here, we will only need explicit formulas for the homomorphism $\varepsilon : \mathcal{O}(\mathrm{SU}_q(2)) \to \mathbb{C}$ given by

$$\varepsilon(\alpha) = \varepsilon(\delta) = 1, \quad \varepsilon(\beta) = \varepsilon(\gamma) = 0.$$

For $s \in (0,1]$, the *-subalgebra generated by

$$\eta_s := (\delta + q^{-1}s\beta)(\beta - s\delta), \quad \zeta_s := 1 - (\alpha - qs\gamma)(\delta + s\beta).$$

is known as the generic Podleś sphere $\mathcal{O}(\mathrm{S}^2_{qs})$ [14]. Its generators satisfy the defining relations

$$\zeta_s\eta_s = q^2\eta_s\zeta_s, \quad \eta_s^*\eta_s = (1-\zeta_s)(s^2+\zeta_s), \quad \eta_s\eta_s^* = (1-q^{-2}\zeta_s)(s^2+q^{-2}\zeta_s),$$

and $\zeta_s^* = \zeta_s$. For all $s \in (0,1]$ and $q \in (0,1)$, the universal C*-algebra of $\mathcal{O}(\mathrm{S}^2_{qs})$ is isomorphic to $\mathcal{C}(\mathrm{S}^2_q)$ [12, 16]. With x the generator of $\mathcal{O}(\mathrm{D}^2_{q^2})$, set $t := 1 - xx* \in \mathcal{T}$. An embedding of $\mathcal{O}(\mathrm{S}^2_{qs})$ into $\mathcal{C}(\mathrm{S}^2_q)$ as a dense *-subalgebra is given by

$$\zeta_s = (-s^2q^2t, q^2t), \quad \eta_s = \Big(s\sqrt{(1-q^2t)(1+s^2q^2t)}\,S\,,\ \sqrt{(1-q^2t)(s^2+q^2t)}\,S\Big). \tag{2.17}$$

Let $\mathcal{O}(\mathrm{S}^2_{qs})^+ := \{x \in \mathcal{O}(\mathrm{S}^2_{qs}) \,:\, \varepsilon(x) = 0\}$. It has been shown in [13] that the quotient space $\mathcal{O}(\mathrm{SU}_q(2))/\mathcal{O}(\mathrm{S}^2_{qs})^+\mathcal{O}(\mathrm{SU}_q(2))$ with coaction $(\mathrm{pr}_s \otimes \mathrm{pr}_s) \circ \Delta$ is a coalgebra isomorphic to $\mathcal{O}(\mathrm{U}(1))$. Here pr_s denotes the canonical projection and Δ the coaction of $\mathcal{O}(\mathrm{SU}_q(2))$. We emphasize that this isomorphism holds only in the category of coalgebras, that is, $\mathcal{O}(\mathrm{SU}_q(2))/\mathcal{O}(\mathrm{S}^2_{qs})^+\mathcal{O}(\mathrm{SU}_q(2))$ is a linear space (not an algebra!) spanned by basis elements U^N, $N \in \mathbb{Z}$, with coaction $\Delta(U^N) = U^N \otimes U^N$. The composition $(\mathrm{id} \otimes \mathrm{pr}_s) \circ \Delta$ turns $\mathcal{O}(\mathrm{SU}_q(2))$ into an $\mathcal{O}(\mathrm{U}(1))$-comodule and the associated line bundles are given by

$$M_N := \{\, p \in \mathcal{O}(\mathrm{SU}_q(2)) \,:\, (\mathrm{id} \otimes \mathrm{pr}_s) \circ \Delta(p) = p \otimes U^N \,\}, \quad N \in \mathbb{Z}.$$

Moreover, $\mathcal{O}(\mathrm{S}^2_{qs}) = M_0 = \mathcal{O}(\mathrm{SU}_q(2))^{\mathrm{co}\mathcal{O}(\mathrm{U}(1))}$ and $\mathcal{O}(\mathrm{SU}_q(2)) = \oplus_{N\in\mathbb{Z}} M_N$. In contrast to quantum line bundles L_N defined above, M_N is only a *left* $\mathcal{O}(\mathrm{S}^2_{qs})$-module but not a *bi*module. This is also due to the fact that $\mathcal{O}(\mathrm{SU}_q(2))$ with above coaction is only an $\mathcal{O}(\mathrm{U}(1))$-comodule but not an $\mathcal{O}(\mathrm{U}(1))$-comodule algebra.

Explicit descriptions of idempotents representing M_N have been given in [6, 15]. Analogous to L_N, there are elements $v^N_0, v^N_1, \ldots, v^N_{|N|} \in \mathcal{O}(\mathrm{SU}_q(2))$ such that $M_N \cong \mathcal{O}(\mathrm{S}^2_{qs})^{|N|+1} P_N$, where

$$P_N := (v^N_0, v^N_1, \ldots, v^N_{|N|})^{\mathrm{t}}\,(v^{N*}_0, v^{N*}_1, \ldots, v^{N*}_{|N|}) \in \mathrm{Mat}_{|N|+1,|N|+1}(\mathcal{O}(\mathrm{S}^2_{qs})) \tag{2.18}$$

with

$$(v^{N*}_0, v^{N*}_1, \ldots, v^{N*}_{|N|})\,(v^N_0, v^N_1, \ldots, v^N_{|N|})^{\mathrm{t}} = 1. \tag{2.19}$$

For a definition of v^N_k, see [15].

A description of the universal C*-algebra $\mathcal{C}(\mathrm{SU}_q(2))$ of $\mathcal{O}(\mathrm{SU}_q(2))$ as a fibre product can be found in [9]. There it is shown that $\mathcal{C}(\mathrm{SU}_q(2))$ is isomorphic to the fibre product C*-algebra of the following pullback diagram:

$$\begin{array}{ccc} & \mathcal{T} \bar{\otimes} \mathcal{C}(\mathrm{S}^1) \underset{(W\circ\sigma\bar{\otimes}\mathrm{id},\pi_2)}{\times} \mathcal{C}(\mathrm{S}^1) & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ \mathcal{T} \bar{\otimes} \mathcal{C}(\mathrm{S}^1) & & \mathcal{C}(\mathrm{S}^1) \\ \downarrow{\scriptstyle \sigma\bar{\otimes}\mathrm{id}} & & \downarrow{\scriptstyle \pi_2} \\ \mathcal{C}(\mathrm{S}^1) \bar{\otimes} \mathcal{C}(\mathrm{S}^1) & \xrightarrow[W]{\qquad\qquad} & \mathcal{C}(\mathrm{S}^1) \bar{\otimes} \mathcal{C}(\mathrm{S}^1)\,. \end{array} \tag{2.20}$$

Here, $\pi_2 : \mathcal{C}(\mathrm{S}^1) \to \mathcal{C}(\mathrm{S}^1) \bar{\otimes} \mathcal{C}(\mathrm{S}^1)$ is defined by $\pi_2(f)(x, y) = f(y)$, and

$$W : \mathcal{C}(\mathrm{S}^1) \bar{\otimes} \mathcal{C}(\mathrm{S}^1) \to \mathcal{C}(\mathrm{S}^1) \bar{\otimes} \mathcal{C}(\mathrm{S}^1), \quad W(f)(x, y) = f(x, xy), \tag{2.21}$$

is the so-called multiplicative unitary. In the next section, we will frequently use that $W(g \otimes U^N)(x, y) = g(x)x^N y^N = (gU^N \otimes U^N)(x, y)$, that is,

$$W(g \otimes U^N) = gU^N \otimes U^N \tag{2.22}$$

for all $g \in \mathcal{C}(\mathrm{S}^1)$ and $N \in \mathbb{Z}$. As above, U denotes the unitary generator of $\mathcal{C}(\mathrm{S}^1)$ given by $U(\mathrm{e}^{\mathrm{i}\phi}) = \mathrm{e}^{\mathrm{i}\phi}$ for $\mathrm{e}^{\mathrm{i}\phi} \in \mathrm{S}^1$.

3. Fibre product approach to quantum Hopf fibrations

3.1. C*-algebraic construction of a quantum Hopf fibration

The aim of this section is to construct a U(1) quantum principal bundle over a quantum 2-sphere such that the associated quantum line bundles are given by (2.11). Our strategy will be to start with trivial U(1)-bundles over two quantum discs and to glue them together along their boundaries by a non-trivial transition function. Working in the category of C*-algebras, an obvious quantum analogue of a trivial bundle $D \times \mathrm{S}^1$ is given by the completed tensor product $\mathcal{T}\,\bar{\otimes}\,\mathcal{C}(\mathrm{S}^1)$, where we regard $\mathcal{T}$ as the algebra of continuous functions on the quantum disc. Since $\mathcal{C}(\mathrm{S}^1)$ is nuclear, there is no ambiguity about the tensor product completion.

Recall from Section 2.1 that a group action on a principal bundle gets translated to a Hopf algebra coaction (or, slightly weaker, coalgebra coaction). As our group is $\mathrm{U}(1) = \mathrm{S}^1$, we take the Hopf *-algebra $\mathcal{C}(\mathrm{S}^1)$ introduced in Section 2.1. On the trivial bundle $\mathcal{T}\,\bar{\otimes}\,\mathcal{C}(\mathrm{S}^1)$, we consider the "trivial" coaction given by applying the coproduct of $\mathcal{C}(\mathrm{S}^1)$ to the second tensor factor. The gluing of the trivial bundles $\mathcal{T}\,\bar{\otimes}\,\mathcal{C}(\mathrm{S}^1)$ will be accomplished by a fibre product over the "boundary" $\mathcal{C}(\mathrm{S}^1)\,\bar{\otimes}\,\mathcal{C}(\mathrm{S}^1)$. To obtain a non-trivial fibre bundle, we impose a non-trivial transition function. From the requirement that the associated quantum line bundles should be given by (2.11), the transition function is easily guessed: We use the multiplicative unitary W from (2.21). The result is described by the following pullback diagram.

$$\begin{array}{ccc} & \mathcal{T}\,\bar{\otimes}\,\mathcal{C}(\mathrm{S}^1) \underset{(W\circ\pi_1,\pi_2)}{\times} \mathcal{T}\,\bar{\otimes}\,\mathcal{C}(\mathrm{S}^1) & \\ {\scriptstyle \mathrm{pr}_1}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_2} \\ \mathcal{T}\,\bar{\otimes}\,\mathcal{C}(\mathrm{S}^1) & & \mathcal{T}\,\bar{\otimes}\,\mathcal{C}(\mathrm{S}^1) \\ {\scriptstyle \pi_1:=\sigma\bar{\otimes}\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \pi_2:=\sigma\bar{\otimes}\mathrm{id}} \\ \mathcal{C}(\mathrm{S}^1)\,\bar{\otimes}\,\mathcal{C}(\mathrm{S}^1) & \xrightarrow[W]{\qquad\qquad} & \mathcal{C}(\mathrm{S}^1)\,\bar{\otimes}\,\mathcal{C}(\mathrm{S}^1)\,. \end{array} \tag{3.1}$$

For brevity, we set $\mathcal{C}(\mathrm{S}^3_q) := \mathcal{T}\,\bar{\otimes}\,\mathcal{C}(\mathrm{S}^1) \times_{(W\circ\pi_1,\pi_2)} \mathcal{T}\,\bar{\otimes}\,\mathcal{C}(\mathrm{S}^1)$. Note that $\sigma\bar{\otimes}\mathrm{id}$ and W are morphisms of right $\mathcal{C}(\mathrm{S}^1)$-comodule algebras. Thus $\mathcal{C}(\mathrm{S}^3_q)$ is a right $\mathcal{C}(\mathrm{S}^1)$-comodule algebra (cf. Section 2.1) or, in the terminology of Section 2.1, a $\mathcal{C}(\mathrm{S}^1)$ quantum principal bundle. Its relation to the (algebraic) Hopf fibration of $\mathcal{O}(\mathrm{S}^3_{pq})$ and to the quantum line bundles from Equation 2.12 will be established in the next proposition.

Proposition 3.1. $\mathcal{C}(\mathrm{S}^3_q)$ *is the universal C*-algebra of* $\mathcal{O}(\mathrm{S}^3_{pq})$*, the associated quantum line bundles*

$$\mathcal{C}(\mathrm{S}^3_q)_N := \{\, p \in \mathcal{C}(\mathrm{S}^3_q) \;:\; \Delta_{\mathcal{C}(\mathrm{S}^3_q)}(p) = p \otimes U^N \,\}, \quad N \in \mathbb{Z}, \tag{3.2}$$

are isomorphic to $\mathcal{T} \times_{(U^N\sigma,\sigma)} \mathcal{T}$ *from* (2.12)*, and* $L_N \subset \mathcal{C}(\mathrm{S}^3_q)_N$*. Here,* L_N *denotes the quantum line bundle defined in* (2.15)*, and* U *is the unitary generator of* $\mathcal{C}(\mathrm{S}^1)$*.*

Proof. Let z and y be the generators of the quantum discs $\mathcal{O}(\mathrm{D}^2_q)$ and $\mathcal{O}(\mathrm{D}^2_p)$, respectively. Consider the *-algebra homomorphism $\iota : \mathcal{O}(\mathrm{S}^3_{pq}) \to \mathcal{C}(\mathrm{S}^3_q)$ given by

$$\iota(a) = (z \otimes U^*, 1 \otimes U^*), \quad \iota(b) = (1 \otimes U, y \otimes U). \tag{3.3}$$

Choosing a Poincaré-Birkhoff-Witt basis of $\mathcal{O}(\mathrm{S}^3_{pq})$, for instance all ordered polynomials in a, a^*, b, b^*, and using the embedding $\mathcal{O}(\mathrm{D}^2_q) \subset \mathcal{T}$, one easily verifies that ι is injective. Moreover, since the operators $\pi(a)$ and $\pi(b)$ satisfy the quantum disc relation (2.7) for any bounded representation π, the *-representation ι is actually an isometry if we equip $\mathcal{O}(\mathrm{S}^3_{pq})$ with the universal C*-norm. Therefore it suffices to prove that $\iota(\mathcal{O}(\mathrm{S}^3_{pq}))$ is dense in $\mathcal{C}(\mathrm{S}^3_q)$. For this, consider the image of $\iota(\mathcal{O}(\mathrm{S}^3_{pq}))$ under the projections pr_1 and pr_2. Since $1 \otimes U = \mathrm{pr}_1(\iota(b)) \in \mathrm{pr}_1(\iota(\mathcal{O}(\mathrm{S}^3_{pq})))$ and $z \otimes 1 = \mathrm{pr}_1(\iota(ab)) \in \mathrm{pr}_1(\iota(\mathcal{O}(\mathrm{S}^3_{pq})))$, we get $\mathrm{pr}_1(\iota(\mathcal{O}(\mathrm{S}^3_{pq}))) = \mathcal{O}(\mathrm{D}^2_q) \otimes \mathcal{O}(\mathrm{U}(1))$, and similarly $\mathrm{pr}_2(\iota(\mathcal{O}(\mathrm{S}^3_{pq}))) = \mathcal{O}(\mathrm{D}^2_q) \otimes \mathcal{O}(\mathrm{U}(1))$. Note that the latter is a dense *-subalgebra of $\mathcal{T} \otimes \mathcal{C}(\mathrm{S}^1)$. Moreover, $(\sigma \bar{\otimes} \mathrm{id})(\mathcal{O}(\mathrm{D}^2_q) \otimes \mathcal{O}(\mathrm{U}(1))) = \mathcal{O}(\mathrm{U}(1)) \otimes \mathcal{O}(\mathrm{U}(1))$ is dense in $\mathcal{C}(\mathrm{S}^1) \bar{\otimes} \mathcal{C}(\mathrm{S}^1)$, and $W : \mathcal{O}(\mathrm{U}(1)) \otimes \mathcal{O}(\mathrm{U}(1)) \to \mathcal{O}(\mathrm{U}(1)) \otimes \mathcal{O}(\mathrm{U}(1))$ is an isometry. Since $W(U^n \otimes U^m) = U^{n+m} \otimes U^m$ for all $n, m \in \mathbb{Z}$ by (2.22), it is a bijection of $\mathcal{O}(\mathrm{U}(1)) \otimes \mathcal{O}(\mathrm{U}(1))$ onto itself. From the foregoing, it follows that $\iota(\mathcal{O}(\mathrm{S}^3_{pq})) = \mathcal{O}(\mathrm{D}^2_q) \otimes \mathcal{O}(\mathrm{U}(1)) \times_{(W \circ \sigma \otimes \mathrm{id}, \pi_2)} \mathcal{O}(\mathrm{D}^2_q) \otimes \mathcal{O}(\mathrm{U}(1))$. By considering the ideal generated by the compact operator $1 - zz^* \in \mathcal{O}(\mathrm{D}^2_q)$ (or $1 - yy^* \in \mathcal{O}(\mathrm{D}^2_p)$), one easily shows that $\ker(\sigma \bar{\otimes} \mathrm{id}) \cap (\mathcal{O}(\mathrm{D}^2_q) \otimes \mathcal{O}(\mathrm{U}(1)))$ is dense in $\ker(\sigma \bar{\otimes} \mathrm{id})$. From the final remark in Section 2.2, we conclude that $\iota(\mathcal{O}(\mathrm{S}^3_{pq}))$ is dense in $\mathcal{C}(\mathrm{S}^3_q)$.

To determine $\mathcal{C}(\mathrm{S}^3_q)_N$, recall that the coaction is given by the coproduct on the second tensor factor $\mathcal{C}(\mathrm{S}^1)$. Assume that $f \in \mathcal{C}(\mathrm{S}^1)$ satisfies $\Delta(f) = f \otimes U^N$. Then it follows from $f = (\varepsilon \otimes \mathrm{id}) \circ \Delta(f) = f(1)U^N$ that $(\mathrm{id} \otimes \Delta)(x) = x \otimes U^N$ for $x \in \mathcal{T} \bar{\otimes} \mathcal{C}(\mathrm{S}^1)$ if and only if $x = t \otimes U^N$ with $t \in \mathcal{T}$. Since the morphisms in the pullback diagram (3.1) are right colinear, we get $p \in \mathcal{C}(\mathrm{S}^3_q)_N$ if and only if $p = (t_1 \otimes U^N, t_2 \otimes U^N)$ and $(W \circ \sigma \bar{\otimes} \mathrm{id})(t_1 \otimes U^N) = (\sigma \bar{\otimes} \mathrm{id})(t_2 \otimes U^N)$. By (2.22), $W(\sigma(t_1) \otimes U^N) = \sigma(t_1)U^N \otimes U^N$. Therefore $(t_1 \otimes U^N, t_2 \otimes U^N) \in \mathcal{C}(\mathrm{S}^3_q)_N$ if and only if $\sigma(t_1)U^N = \sigma(t_2)$. This shows that an isomorphism between $\mathcal{C}(\mathrm{S}^3_q)_N$ and $\mathcal{T} \times_{(U^N\sigma, \sigma)} \mathcal{T}$ is given by

$$\mathcal{C}(\mathrm{S}^3_q)_N \ni (t_1 \otimes U^N, t_2 \otimes U^N) \mapsto (t_1, t_2) \in \mathcal{T} \times_{(U^N\sigma, \sigma)} \mathcal{T}. \tag{3.4}$$

From (3.3) and $\Delta(U^N) = U^N \otimes U^N$, it follows that $\Delta_{\mathcal{C}(\mathrm{S}^3_q)}(\iota(a)) = \iota(a) \otimes U^*$ and $\Delta_{\mathcal{C}(\mathrm{S}^3_q)}(\iota(b)) = \iota(b) \otimes U$. Hence ι is right colinear. Since ι is also an isometry, we can view $\mathcal{O}(\mathrm{S}^3_{pq})$ as a subalgebra of $\mathcal{C}(\mathrm{S}^3_q)$. Then $L_N \subset \mathcal{C}(\mathrm{S}^3_q)_N$ by the definitions of L_N and $\mathcal{C}(\mathrm{S}^3_q)_N$ in (2.15) and (3.2), respectively. □

We remark that the universal C*-algebra of $\mathcal{O}(\mathrm{S}^3_{pq})$ has been studied in [7], the K-theory of $\mathcal{C}(\mathrm{S}^3_q)$ has been determined in [1]; and from the last example in [5], it follows that $\mathcal{C}(\mathrm{S}^3_q)$ behaves well under the $\mathcal{C}(\mathrm{S}^1)$-coaction (it is a principal Hopf-Galois extension).

3.2. Index computation for quantum line bundles

The aim of this section is to illustrate that the fibre product approach may lead to a significant simplification of index computations. First we remark that, in (algebraic) quantum group theory, algebras are frequently defined by generators and relations similar those in (2.14) for $\mathcal{O}(\mathrm{S}^3_{pq})$ (more examples can be found, e.g., in [11]). A pair of *-representations on the same Hilbert space such that the difference yields compact operators gives rise to an even Fredholm module and can be used for index computations by pairing it with K_0-classes. If we want to compute for instance the index pairing with the K_0-class of the projective modules L_N from (2.15) by using the idempotents given in (2.16), then we face difficulties because of the growing size of the matrices. It is therefore desirable to find simpler representatives of K-theory classes of the projective modules L_N. This section shows that the fibre product approach provides us with an effective tool for obtaining more suitable projections. In our example, the index pairing will reduce to its simplest possible form: it remains to calculate a trace of a projection onto a finite-dimensional subspace.

We start by proving that the projective modules $\mathcal{C}(\mathrm{S}^3_{pq})_N$ can be represented by elementary one-dimensional projections. Because of the isomorphism between $\mathcal{C}(\mathrm{S}^3_{pq})_N$ and $\mathcal{T} \times_{(U^N\sigma,\sigma)} \mathcal{T}$ in Proposition 3.1, this result has already been obtained in [17]. For the convenience of the reader, we include here the proof. It uses essentially the same "bra-ket" argument that was used in [6, 15] to prove $M_N \cong \mathcal{O}(\mathrm{S}^2_{qs})^{|N|+1}P_N$ for the Hopf fibration of $\mathcal{O}(\mathrm{SU}_q(2))$.

Proposition 3.2. *For $N \in \mathbb{Z}$, define*

$$\chi_N := (1\,,\, S^{|N|}S^{*|N|}) \in \mathcal{C}(\mathrm{S}^2_q), \text{ for } N < 0, \tag{3.5}$$

$$\chi_N := (S^N S^{*N}\,,\, 1) \in \mathcal{C}(\mathrm{S}^2_q), \quad \text{for } N \geq 0. \tag{3.6}$$

Then the left $\mathcal{C}(\mathrm{S}^2_q)$-modules $\mathcal{C}(\mathrm{S}^3_{pq})_N$ and $\mathcal{C}(\mathrm{S}^2_q)\chi_N$ are isomorphic.

Proof. Since $\sigma(S^nS^{*n}) = U^nU^{*n} = 1$ for all $n \in \mathbb{N}$, the projections χ_N belong to $\mathcal{C}(\mathrm{S}^2_q) = \mathcal{T} \times_{(\sigma,\sigma)} \mathcal{T}$. We will use the isomorphism of Proposition 3.1 and prove that $\mathcal{C}(\mathrm{S}^2_q)\chi_N$ is isomorphic to $\mathcal{E}_N := \mathcal{T} \times_{(U^N\sigma,\sigma)} \mathcal{T}$.

Let $N \geq 0$. From (2.10) and (2.12), it follows that $(fS^{*N}\,,\, g) \in \mathcal{C}(\mathrm{S}^2_q)$ for all $(f,g) \in \mathcal{E}_N$. Therefore we can define a $\mathcal{C}(\mathrm{S}^2_q)$-linear map $\Psi_N : \mathcal{E}_N \to \mathcal{C}(\mathrm{S}^2_q)\chi_N$ by

$$\Psi_N(f,g) := (fS^{*N}\,,\, g)\chi_N = (fS^{*N}\,,\, g), \tag{3.7}$$

where we used $S^*S = \mathrm{id}$ in the second equality. Since S^* is right invertible, we have $(fS^{*N}\,,\, g) = 0$ if and only if $(f\,,\, g) = 0$, hence Ψ_N is injective.

Now let $(f,g)\chi_N \in \mathcal{C}(\mathrm{S}^2_q)\chi_N$. Then $(fS^N, g) \in \mathcal{E}_N$ and $\Psi_N(fS^N, g) = (fS^NS^{*N}, g) = (f,g)\chi_N$, thus Ψ_N is also surjective. This proves the claim of Proposition 3.2 for $N \geq 0$. The proof for $N < 0$ runs analogously with Ψ_N defined by $\Psi_N(f,g) := (f\,,\, gS^{*N})\chi_N$. □

Clearly, the (left) multiplication by elements of the C*-algebra $\mathcal{C}(\mathrm{S}^2_q)$ turns $L_N \cong \mathcal{O}(\mathrm{S}^2_{pq})^{|N|+1}E_N$ into a (left) $\mathcal{C}(\mathrm{S}^2_q)$-module. With a slight abuse of notation, we set $\mathcal{C}(\mathrm{S}^2_q)L_N := \mathrm{span}\{xv : x \in \mathcal{C}(\mathrm{S}^2_q),\ v \in L_N\}$. (Later it turns out that this module is generated by one element in L_N so that the notation is actually correct.) If we show that $\mathcal{C}(\mathrm{S}^2_q)L_N$ is isomorphic to $\mathcal{C}(\mathrm{S}^3_{pq})_N$, then the elementary projections χ_N and the $(|N|+1)\times(|N|+1)$-matrices E_N define the same K_0-class. The desired isomorphism will be established in the next proposition by using the embedding $L_N \subset \mathcal{C}(\mathrm{S}^3_{pq})_N$ from Proposition 3.1.

Proposition 3.3. *The left $\mathcal{C}(\mathrm{S}^2_q)$-modules $\mathcal{C}(\mathrm{S}^2_q)L_N \cong \mathcal{C}(\mathrm{S}^2_q)^{|N|+1}E_N$ and $\mathcal{C}(\mathrm{S}^3_{pq})_N$ are isomorphic.*

Proof. Using embedding (3.3) and the inclusion from Proposition 3.1, we can view $\mathcal{C}(\mathrm{S}^2_q)L_N = \mathrm{span}\{xv : x \in \mathcal{C}(\mathrm{S}^2_q),\ v \in L_N\} \subset \mathcal{C}(\mathrm{S}^3_{pq})_N$ as a submodule of $\mathcal{C}(\mathrm{S}^3_{pq})_N$. Let $N \in \mathbb{N}_0$. It follows from the isomorphism Ψ_N defined in (3.7) that the left $\mathcal{C}(\mathrm{S}^2_q)$-module $\mathcal{C}(\mathrm{S}^3_{pq})_N = \{(fS^{*N}, g) : (f,g) \in \mathcal{C}(\mathrm{S}^2_q)\}$ is generated by the element $(S^{*N}, 1)$. Therefore, to prove $\mathcal{C}(\mathrm{S}^2_q)L_N = \mathcal{C}(\mathrm{S}^3_{pq})_N$, it suffices to show that $(S^{*N}, 1) \in \mathcal{C}(\mathrm{S}^2_q)L_N$. Since $\sigma(z^{*n}) = U^{-N}$, we have $(z^{*N}, 1) \in \mathcal{T} \times_{(U^N\sigma,\sigma)} \mathcal{T}$. Since $(z^{*N}, 1)$ is the image of $\iota(a^{*N}) = (z^{*N} \otimes U^N, 1 \otimes U^N)$ under the isomorphism (3.4), we can view $(z^{*N}, 1)$ as an element of L_N. Let $t := 1 - zz^* \in \mathcal{T}$. Note that t is a self-adjoint operator with spectrum $\mathrm{spec}(t) = \{q^n : n \in \mathbb{N}_0\} \cup \{0\}$ (see Equation (2.8)). Applying the commutation relations (2.7), one easily verifies that $z^{*N}z^N = \Pi_{k=1}^N(1 - q^k t)$. Since $\mathrm{spec}(t) \subset [0,1]$, the operator $z^{*N}z^N$ is strictly positive. Hence $|z^N|^{-1} = (z^{*N}z^N)^{-1/2}$ belongs to the C*-algebra $\mathcal{T}$. Moreover, $\sigma(|z^N|^{-1}) = 1$ since $\sigma(z^{*N}z^N) = 1$. Therefore $(|z^N|^{-1}, 1) \in \mathcal{T} \times_{(\sigma,\sigma)} \mathcal{T} = \mathcal{C}(\mathrm{S}^2_q)$ and thus $(S^{*N}, 1) = (|z^N|^{-1}, 1)(z^{*N}, 1) \in \mathcal{C}(\mathrm{S}^2_q)L_N$. This completes the proof for $N \geq 0$. The case $N < 0$ is treated analogously. □

Recall that an (even) Fredholm module of an *-algebra $\mathcal{A}$ can be given by a pair of *-representations (ρ_+, ρ_-) of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ such that the difference $\rho_+(a) - \rho_-(a)$ yields a compact operator. In this case, for any projection $P \in \mathrm{Mat}_{n,n}(\mathcal{A})$, the operator $\varrho_+(P)\varrho_-(P) : \varrho_-(P)\mathcal{H}^n \to \varrho_+(P)\mathcal{H}^n$ is a Fredholm operator and its Fredholm index does neither depend on the K_0-class of P nor on the class of (ρ_+, ρ_-) in K-homology. This pairing between K-theory and K-homology is referred to as index pairing. If it happens that $\rho_+(a) - \rho_-(a)$ yields trace class operators, then the index pairing can be computed by a trace formula, namely

$$\langle [(\rho_+, \rho_-)], [P] \rangle = \mathrm{tr}_{\mathcal{H}}(\mathrm{tr}_{\mathrm{Mat}_{n,n}}(\rho_+ - \rho_-)(P)) \tag{3.8}$$

In general, the computation of the traces gets more involved with increasing size of the matrix P. This will especially be the case if one works only with the polynomial algebras $\mathcal{O}(\mathrm{S}^3_{pq})$ and $\mathcal{O}(\mathrm{S}^2_{pq})$, and uses the $(|N|+1)\times(|N|+1)$-projections E_N from (2.16) with entries in belonging to $\mathcal{O}(\mathrm{S}^2_{pq})$. In our example, the C*-algebraic fibre product approach improves the situation considerably since Propositions 3.2 and 3.3 provide us with the equivalent one-dimensional projections χ_N. As the index computation is one of our main objectives, we state the result in the following theorem.

Theorem 3.4. *Let $N \in \mathbb{Z}$. The isomorphic projective left $\mathcal{C}(\mathrm{S}_q^2)$-modules $\mathcal{C}(\mathrm{S}_{pq}^3)_N$, $\mathcal{C}(\mathrm{S}_q^2)L_N$, $\mathcal{C}(\mathrm{S}_q^2)^{|N|+1}E_N$ and $\mathcal{C}(\mathrm{S}_q^2)\chi_N$ define the same class in $K_0(\mathcal{C}(\mathrm{S}_q^2))$, say $[\chi_N]$, and the pairing with the generators of the K-homology $K^0(\mathcal{C}(\mathrm{S}_q^2))$ from the end of Section 2.3 is given by*

$$\langle\, [(\mathrm{pr}_1, \mathrm{pr}_0)]\, , [\chi_N]\, \rangle = N, \qquad \langle\, [(\pi_+ \circ \sigma\, ,\, \pi_- \circ \sigma)]\, , [\chi_N]\, \rangle = 1. \tag{3.9}$$

Proof. The equivalences of the left $\mathcal{C}(\mathrm{S}_q^2)$-modules has been shown in Propositions 3.2 and 3.3. In particular, we are allowed to choose χ_N as a representative.

For all $N \in \mathbb{Z}$, the operator $\pi_+ \circ \sigma(\chi_N) - \pi_- \circ \sigma(\chi_N) = \pi_+(1) - \pi_-(1)$ is the projector onto the one-dimensional subspace $\mathbb{C}e_0$, see Equation (2.13). In particular, it is of trace class so that Equation (3.8) applies. Since the trace of a one-dimensional projection is 1, we get

$$\langle\, [(\pi_+ \circ \sigma\, ,\, \pi_- \circ \sigma)]\, , [\chi_N]\, \rangle = \mathrm{tr}_{\ell^2(\mathbb{N}_0)}(\pi_+(1) - \pi_-(1)) = 1.$$

Now let $N \geq 0$. Then $(\mathrm{pr}_1, \mathrm{pr}_0)(\chi_N) = (\mathrm{pr}_1 - \mathrm{pr}_0)(S^N S^{*N}, 1) = 1 - S^N S^{*N}$ is the projection onto the subspace $\mathrm{span}\{e_0, \ldots, e_{n-1}\}$. Since it is of trace class with trace equal to the dimension of its image, we can apply Equation (3.8) and get

$$\langle\, [(\mathrm{pr}_1, \mathrm{pr}_0)]\, , [\chi_N]\, \rangle = \mathrm{tr}_{\ell^2(\mathbb{N}_0)}(1 - S^N S^{*N}) = N.$$

Analogously, for $N < 0$,

$$\langle\, [(\mathrm{pr}_1, \mathrm{pr}_0)]\, , [\chi_N]\, \rangle = \mathrm{tr}_{\ell^2(\mathbb{N}_0)}(S^{|N|} S^{*|N|} - 1) = -|N| = N,$$

which completes the proof. □

Since the C*-algebra $\mathcal{C}(\mathrm{S}_q^2)$ is isomorphic to the universal C*-algebra of the Podleś spheres $\mathcal{O}(\mathrm{S}_{qs}^2)$, the indices in Equation (3.9) have also been obtained in [6] and [17]. In the first paper, the computations relied heavily on the index theorem, whereas in [17] and Theorem 3.4, by using the fibre product approach, the traces were computed directly by finding equivalent elementary projections.

Note that Equation (3.9) has a geometrical interpretation: The pairing with the K-homology class $[(\pi_+ \circ \sigma\, ,\, \pi_- \circ \sigma)]$ detects the rank of the projective module, and the pairing $\langle\, [(\mathrm{pr}_1, \mathrm{pr}_0)]\, , [\chi_N]\, \rangle = N$ coincides with the power of U in (2.11) and thus computes the "winding number", that is, the number of rotations of the transition function along the equator.

3.3. Equivalence to the generic Hopf fibration of quantum SU(2)

Recall from Section 2.4 that $\mathcal{O}(\mathrm{SU}_q(2)) = \oplus_{N\in\mathbb{Z}} M_N$, where

$$M_N := \{\, p \in \mathcal{O}(\mathrm{SU}_q(2))\, :\, \Delta_{\mathcal{O}(\mathrm{SU}_q(2))}(p) = p \otimes U^N\, \} \cong \mathcal{O}(\mathrm{S}_{qs}^2)^{|N|+1} P_N$$

with $P_N \in \mathrm{Mat}_{|N|+1,|N|+1}(\mathcal{O}(\mathrm{S}_{qs}^2))$ given in Equation (2.18). For the definition of the $\mathcal{O}(\mathrm{U}(1))$-coaction $\Delta_{\mathcal{O}(\mathrm{SU}_q(2))} = (\mathrm{id} \otimes \mathrm{pr}_s) \circ \Delta$, see Section 2.4. Since $\mathcal{O}(\mathrm{S}_{qs}^2)$ can be embedded into its universal C*-algebra, which is isomorphic to $\mathcal{C}(\mathrm{S}_q^2)$, we can turn M_N into a left $\mathcal{C}(\mathrm{S}^2_q)$-module by considering $\overline{M}_N := \mathcal{C}(\mathrm{S}^2_q)^{|N|+1}P_N$. It has been shown in [17], that this left $\mathcal{C}(\mathrm{S}^2_q)$-module is isomorphic to $\mathcal{C}(\mathrm{S}^2_q)\chi_N$, and therefore to $\mathcal{C}(\mathrm{S}^3_{pq})_N$.

The aim of this section is to define a left $\mathcal{C}(\mathrm{S}^2_q)$-module and right $\mathcal{O}(\mathrm{U}(1))$-comodule P such that, for all $N \in \mathbb{Z}$, the line bundle associated to the 1-dimensional left corepresentation ${}_{\mathbb{C}}\Delta(1) = U^N \otimes 1$ is isomorphic to $\overline{M}_N$. A natural idea would be to consider the embedding of $\mathcal{O}(\mathrm{SU}_q(2))$ into $\mathcal{C}(\mathrm{SU}_q(2))$ and to extend the right coaction $\Delta_{\mathcal{O}(\mathrm{SU}_q(2))}$ to the C*-algebra closure. But then we face the problem that $\Delta_{\mathcal{O}(\mathrm{SU}_q(2))}$ is merely a coaction and not an algebra homomorphism. If we impose at $\mathcal{O}(\mathrm{U}(1))$ the obvious multiplicative structure given by $U^N U^K = U^{N+K}$, and turn $\oplus_{N\in\mathbb{Z}}\overline{M}_N$ into a *-algebra such that the right $\mathcal{O}(\mathrm{U}(1))$-coaction becomes an algebra homomorphism, then the C*-closure of $\oplus_{N\in\mathbb{Z}}\overline{M}_N \cong \oplus_{N\in\mathbb{Z}}\mathcal{C}(\mathrm{S}^3_{pq})_N$ would be isomorphic to $\mathcal{C}(\mathrm{S}^3_{pq})$ and not to $\mathcal{C}(\mathrm{SU}_q(2))$. Note that there cannot be an isomorphism between $\mathcal{C}(\mathrm{S}^3_{pq})$ and $\mathcal{C}(\mathrm{SU}_q(2))$ since otherwise, by the pullback diagrams (2.20) and (3.1), $\mathcal{C}(\mathrm{S}^1) \cong \ker(\mathrm{pr}_1) \cong \mathcal{T}\bar{\otimes}\mathcal{C}(\mathrm{S}^1)$, a contradiction.

Instead of extending the coaction $\Delta_{\mathcal{O}(\mathrm{SU}_q(2))}$ to some closure of $\mathcal{O}(\mathrm{SU}_q(2))$, we turn $\mathcal{O}(\mathrm{SU}_q(2))$ into a left $\mathcal{C}(\mathrm{S}^2_q)$-module by setting $P = \mathcal{C}(\mathrm{S}^2_q) \otimes_{\mathcal{O}(\mathrm{S}^2_{qs})} \mathcal{O}(\mathrm{SU}_q(2))$ and keeping the $\mathcal{O}(\mathrm{U}(1))$-coaction, now acting on the second tensor factor. Then it follows immediately that

$$P = \bigoplus_{N\in\mathbb{Z}} \mathcal{C}(\mathrm{S}^2_q) \underset{\mathcal{O}(\mathrm{S}^2_{qs})}{\otimes} M_N \quad\text{and}\quad \mathcal{C}(\mathrm{S}^2_q) \underset{\mathcal{O}(\mathrm{S}^2_{qs})}{\otimes} M_N = \{p \in P \,:\, \Delta_P(p) = p \otimes U^N\}.$$

Thus our aim will be achieved if we show that $\overline{M}_N \cong \mathcal{C}(\mathrm{S}^2_q) \otimes_{\mathcal{O}(\mathrm{S}^2_{qs})} M_N$. For this, we prove that P, as a left $\mathcal{C}(\mathrm{S}^2_q)$-module and right $\mathcal{O}(\mathrm{U}(1))$-comodule, is isomorphic to the following fibre product

$$\begin{array}{ccc} & \mathcal{T}\otimes\mathcal{O}(\mathrm{U}(1)) \underset{(\Phi\circ\pi_1,\pi_2)}{\times} \mathcal{T}\otimes\mathcal{O}(\mathrm{U}(1)) & \\ \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} \\ \mathcal{T}\otimes\mathcal{O}(\mathrm{U}(1)) & & \mathcal{T}\otimes\mathcal{O}(\mathrm{U}(1)) \\ \downarrow{\scriptstyle \pi_1:=\sigma\otimes\mathrm{id}} & & \downarrow{\scriptstyle \pi_2:=\sigma\otimes\mathrm{id}} \\ \mathcal{C}(\mathrm{S}^1)\otimes\mathcal{O}(\mathrm{U}(1)) & \xrightarrow[\Phi]{\qquad\qquad} & \mathcal{C}(\mathrm{S}^1)\otimes\mathcal{O}(\mathrm{U}(1)) . \end{array} \tag{3.10}$$

Here Φ is defined by $\Phi(f \otimes U^N) = fU^N \otimes U^N$. Then, by comparing the pullback diagrams (3.1) and (3.10) in the category of left $\mathcal{C}(\mathrm{S}^2_q)$-modules and right $\mathcal{C}(\mathrm{S}^1)$-comodules, it follows that

$$\overline{M}_N \cong \mathcal{C}(\mathrm{S}^3_{pq})_N \cong P \,\Box_{\mathcal{C}(\mathrm{S}^2_q)}\, \mathbb{C} \cong \mathcal{C}(\mathrm{S}^2_q) \otimes_{\mathcal{O}(\mathrm{S}^2_{qs})} M_N \tag{3.11}$$

with ${}_{\mathcal{O}(\mathrm{U}(1))}\Delta(1) = U^N \otimes 1$ on $\mathbb{C}$ in the cotensor product.

For simplicity of notation, we set

$$\mathcal{A} := \mathcal{O}(\mathrm{SU}_q(2)), \quad \mathcal{B} := \mathcal{O}(\mathrm{S}^2_{qs}), \quad \overline{\mathcal{B}} := \mathcal{T} \underset{(\sigma,\sigma)}{\times} \mathcal{T} \cong \mathcal{C}(\mathrm{S}^2_q), \quad C := \mathcal{O}(\mathrm{U}(1)).$$

Recall that $\mathcal{B}$ can be embedded in $\mathcal{A}$ as well as in $\overline{\mathcal{B}}$, so both are $\mathcal{B}$-bimodules with respect to the multiplication. Moreover, the pullback diagram (2.9) provides us with *-algebra homomorphism $\mathrm{pr}_0 : \overline{\mathcal{B}} \to \mathcal{T}$ and $\mathrm{pr}_1 : \overline{\mathcal{B}} \to \mathcal{T}$ by projecting onto the left and right component, respectively. Perhaps it should here also be mentioned that C is only considered as a coalgebra, not as an algebra.

Let $v_0^N, v_1^N, \ldots, v_{|N|}^N \in \mathcal{A}$ denote the matrix elements from the definition of P_N in (2.18). Since the entries of P_N belong to $\mathcal{B}$, we have $v_j^N v_k^{*N} \in \mathcal{B}$ for all $j, k = 0, \ldots, |N|$. The following facts are proven in [15, Lemma 7.5].

Lemma 3.5. *Let $l \in \mathbb{Z}$ and $k, m \in \{0, \ldots |l|\}$.*

(i) *For $l \geq 0$, the elements $\mathrm{pr}_1(v_l^l v_l^{l*})$ and $\mathrm{pr}_0(v_0^l v_0^{l*})$ are invertible in $\mathcal{T}$.*

(ii) *For $l < 0$, the elements $\mathrm{pr}_1(v_0^l v_0^{l*})$ and $\mathrm{pr}_0(v_{|l|}^l v_{|l|}^{l*})$ are invertible in $\mathcal{T}$.*

(iii) $\mathrm{pr}_1(v_k^l v_l^{l*})\,\mathrm{pr}_1(v_l^l v_l^{l*})^{-1}\,\mathrm{pr}_1(v_l^l v_m^{l*}) = \mathrm{pr}_1(v_k^l v_m^{l*})$ *and* $\mathrm{pr}_0(v_k^l v_0^{l*})\mathrm{pr}_0(v_0^l v_0^{l*})^{-1}\mathrm{pr}_0(v_0^l v_m^{l*}) = \mathrm{pr}_0(v_k^l v_m^{l*})$ *for $l \geq 0$.*

(iv) $\mathrm{pr}_1(v_k^l v_0^{l*})\,\mathrm{pr}_1(v_0^l v_0^{l*})^{-1}\,\mathrm{pr}_1(v_0^l v_m^{l*}) = \mathrm{pr}_1(v_k^l v_m^{l*})$ *and* $\mathrm{pr}_0(v_k^l v_{|l|}^{l*})\mathrm{pr}_0(v_{|l|}^l v_{|l|}^{l*})^{-1}\mathrm{pr}_0(v_{|l|}^l v_m^{l*}) = \mathrm{pr}_0(v_k^l v_m^{l*})$ *for $l < 0$,*

We can turn $\mathcal{T}$ into a $\overline{\mathcal{B}}$-bimodule by setting $a.t.b := \mathrm{pr}_0(a)\, t\, \mathrm{pr}_0(b)$ and $a.t.b = \mathrm{pr}_1(a)\, t\, \mathrm{pr}_1(b)$, where $a, b \in \overline{\mathcal{B}}$ and $t \in \mathcal{T}$. To distinguish between both bimodules, we denote $\mathcal{T}$ equipped with the first action by $\mathcal{T}_-$, and write $\mathcal{T}_+$ if we use the second action. Clearly, as left or right $\overline{\mathcal{B}}$-module, both are generated by $1 \in \mathcal{T}$. The next proposition is the key in proving (3.11).

Proposition 3.6. *The left $\overline{\mathcal{B}}$-modules $\mathcal{T}_\pm$ and $\mathcal{T}_\pm \otimes_{\mathcal{B}} M_l$ are isomorphic. The corresponding isomorphisms are given by*

$$\psi_{l,+} : \mathcal{T}_+ \to \mathcal{T}_+ \otimes_{\mathcal{B}} M_l, \quad \psi_{l,+}(t) = t\, \mathrm{pr}_0(v_0^l v_0^{l*})^{-1/2} \otimes_{\mathcal{B}} v_0^l, \qquad l \geq 0,$$
$$\psi_{l,-} : \mathcal{T}_- \to \mathcal{T}_- \otimes_{\mathcal{B}} M_l, \quad \psi_{l,-}(t) = t\, \mathrm{pr}_1(v_l^l v_l^{l*})^{-1/2} \otimes_{\mathcal{B}} v_l^l, \qquad l \geq 0,$$
$$\psi_{l,+} : \mathcal{T}_+ \to \mathcal{T}_+ \otimes_{\mathcal{B}} M_l, \quad \psi_{l,+}(t) = t\, \mathrm{pr}_0(v_{|l|}^l v_{|l|}^{l*})^{-1/2} \otimes_{\mathcal{B}} v_{|l|}^l, \quad l < 0,$$
$$\psi_{l,-} : \mathcal{T}_- \to \mathcal{T}_- \otimes_{\mathcal{B}} M_l, \quad \psi_{l,-}(t) = t\, \mathrm{pr}_1(v_0^l v_0^{l*})^{-1/2} \otimes_{\mathcal{B}} v_0^l, \qquad l < 0.$$

The inverse isomorphisms satisfy, for all $k = 0, 1, \ldots, |l|$,

$$\psi_{l,+}^{-1}(1 \otimes_{\mathcal{B}} v_k^l) = \mathrm{pr}_0(v_k^l v_0^{l*})\mathrm{pr}_0(v_0^l v_0^{l*})^{-1/2}, \qquad l \geq 0, \tag{3.12}$$
$$\psi_{l,-}^{-1}(1 \otimes_{\mathcal{B}} v_k^l) = \mathrm{pr}_1(v_k^l v_l^{l*})\mathrm{pr}_1(v_l^l v_l^{l*})^{-1/2}, \qquad l \geq 0, \tag{3.13}$$
$$\psi_{l,+}^{-1}(1 \otimes_{\mathcal{B}} v_k^l) = \mathrm{pr}_1(v_k^l v_{|l|}^{l*})\,\mathrm{pr}_1(v_{|l|}^l v_{|l|}^{l*})^{-1/2}, \quad l < 0, \tag{3.14}$$
$$\psi_{l,-}^{-1}(1 \otimes_{\mathcal{B}} v_k^l) = \mathrm{pr}_1(v_k^l v_0^{l*})\,\mathrm{pr}_1(v_0^l v_0^{l*})^{-1/2}, \quad l < 0. \tag{3.15}$$

Proof. We prove the proposition for $\psi_{l,+}$ with $l \geq 0$, the other cases are treated analogously. Since $\mathrm{pr}_0(v_0^l v_0^{l*})$ is positive and invertible, $\mathrm{pr}_0(v_0^l v_0^{l*})^{-1/2} \in \mathcal{T}$ is invertible, and thus $\psi_{l,+}$ is injective. The left $\overline{\mathcal{B}}$-module $\mathcal{T}_+ \otimes_{\mathcal{B}} M_l$ is generated by $1 \otimes_{\mathcal{B}} v_k^l$, $k = 0, 1, \ldots, l$ (cf. [15, Theorem 4.1]). As $\psi_{l,+}$ is left $\overline{\mathcal{B}}$-linear, it suffices to prove that the elements $1 \otimes_{\mathcal{B}} v_k^l$ belong to the image of $\psi_{l,+}$. Applying (2.19) and Lemma 3.5(iii), we get

$$\begin{aligned}
1 \otimes_{\mathcal{B}} v_k^l &= \textstyle\sum_j 1 \otimes_{\mathcal{B}} v_k^l v_j^{l*} v_j^l \\
&= \textstyle\sum_j \mathrm{pr}_0(v_k^l v_j^{l*}) \otimes_{\mathcal{B}} v_j^l \\
&= \textstyle\sum_j \mathrm{pr}_0(v_k^l v_0^{l*}) \mathrm{pr}_0(v_0^l v_0^{l*})^{-1} \mathrm{pr}_0(v_0^l v_j^{l*}) \otimes_{\mathcal{B}} v_j^l \\
&= \textstyle\sum_j \mathrm{pr}_0(v_k^l v_0^{l*}) \mathrm{pr}_0(v_0^l v_0^{l*})^{-1} \otimes_{\mathcal{B}} v_0^l v_j^{l*} v_j^l \\
&= \mathrm{pr}_0(v_k^l v_0^{l*}) \mathrm{pr}_0(v_0^l v_0^{l*})^{-1} \otimes_{\mathcal{B}} v_0^l \\
&= \psi_{l,+}\big(\mathrm{pr}_0(v_k^l v_0^{l*}) \mathrm{pr}_0(v_0^l v_0^{l*})^{-1/2}\big).
\end{aligned}$$

This proves the surjectivity of $\psi_{l,+}$ and Equation (3.12). □

Using the last proposition and the decomposition $\mathcal{A} = \oplus_{N \in \mathbb{Z}} M_N$, we can define left $\overline{\mathcal{B}}$-linear, right C-colinear isomorphisms

$$\Psi_- : \mathcal{T}_- \otimes_{\mathcal{B}} \mathcal{A} \to \bigoplus_{N \in \mathbb{Z}} \mathcal{T}_- \otimes U^N, \quad \Psi_+ : \mathcal{T}_+ \otimes_{\mathcal{B}} \mathcal{A} \to \bigoplus_{N \in \mathbb{Z}} \mathcal{T}_+ \otimes U^N$$

by setting

$$\Psi_\pm(t \otimes_{\mathcal{B}} m_N) = \psi_{N,\pm}^{-1}(t \otimes_{\mathcal{B}} m_N) \otimes U^N, \qquad t \in \mathcal{T}, \ \ m_N \in M_N. \tag{3.16}$$

Next we define left $\overline{\mathcal{B}}$-linear, right C-colinear surjections

$$\mathrm{pr}_\pm : \overline{\mathcal{B}} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{T}_\pm \otimes_{\mathcal{B}} \mathcal{A},$$

by

$$\mathrm{pr}_-((t_1, t_2) \otimes_{\mathcal{B}} a) := t_1 \otimes_{\mathcal{B}} a, \quad \mathrm{pr}_+((t_1, t_2) \otimes_{\mathcal{B}} a) := t_2 \otimes_{\mathcal{B}} a. \tag{3.17}$$

Furthermore, we turn $\mathcal{C}(\mathrm{S}^1)$ into a left $\overline{\mathcal{B}}$-module by defining $b.f := \sigma(b) f$ for all $b \in \overline{\mathcal{B}}$ and $f \in \mathcal{C}(\mathrm{S}^1)$. Now consider the following diagram in the category of left $\overline{\mathcal{B}}$-modules, right C-comodules:

$$\begin{array}{ccc}
\overline{\mathcal{B}} \otimes_{\mathcal{B}} \mathcal{A} & \xrightarrow{\Psi_+ \circ \mathrm{pr}_+} & \mathcal{T}_+ \otimes C \\
{\scriptstyle \Psi_- \circ \mathrm{pr}_-} \big\downarrow & & \big\downarrow {\scriptstyle \sigma \otimes \mathrm{id}} \\
\mathcal{T}_- \otimes C & \xrightarrow[\Phi \circ (\sigma \otimes \mathrm{id})]{} & \mathcal{C}(\mathrm{S}^1) \otimes C,
\end{array} \tag{3.18}$$

where Φ is the same as in (3.10).

Lemma 3.7. *The diagram* (3.18) *is commutative,* $\Psi_- \circ \mathrm{pr}_-$ *and* $\Psi_+ \circ \mathrm{pr}_+$ *are surjective and* $\ker(\Psi_- \circ \mathrm{pr}_-) \cap \ker(\Psi_+ \circ \mathrm{pr}_+) = \{0\}$.

Proof. Since all maps are left $\overline{\mathcal{B}}$-linear, it suffices to prove the lemma for generators of the left $\overline{\mathcal{B}}$-module $\overline{\mathcal{B}} \otimes_{\mathcal{B}} \mathcal{A}$. Moreover, since $\mathcal{A} = \oplus_{N\in\mathbb{Z}} M_N$, we can restrict ourselves to the generators of the left $\mathcal{B}$-modules M_N.

Let $l \geq 0$. Since $\sigma(\mathrm{pr}_0(f)) = \sigma(f) = \sigma(\mathrm{pr}_1(f))$ for all $f \in \overline{\mathcal{B}}$ by (2.10), we get from Equation (3.16) and Lemma 3.5

$$(\sigma \otimes \mathrm{id}) \circ \Psi_+ \circ \mathrm{pr}_-\big(1 \otimes_{\mathcal{B}} v^l_k\big) = \sigma(v^l_k v^{l*}_0)\sigma(v^l_0 v^{l*}_0)^{-1/2} \otimes U^l, \tag{3.19}$$

$$\phi \circ (\sigma \otimes \mathrm{id}) \circ \Psi_- \circ \mathrm{pr}_+\big(1 \otimes_{\mathcal{B}} v^l_k\big) = \sigma(v^l_k v^{l*}_l)\sigma(v^l_l v^{l*}_l)^{-1/2} U^l \otimes U^l. \tag{3.20}$$

By Lemma 3.5 (iii) (with $m = 0$), we have

$$\sigma(v^l_k v^{l*}_0) = \sigma(v^l_k v^{l*}_l)\, \sigma(v^l_l v^{l*}_l)^{-1}\, \sigma(v^l_l v^{l*}_0). \tag{3.21}$$

Inserting the latter equation into (3.19) and comparing with (3.20) shows that it suffices to prove

$$\sigma(v^l_l v^{l*}_l)^{-1/2}\, \sigma(v^l_l v^{l*}_0)\, \sigma(v^l_0 v^{l*}_0)^{-1/2} = U^l. \tag{3.22}$$

It follows from [17, Lemma 2.2] (with $v^l_l \sim u_l$ and $v^l_0 \sim w_l$), or can be computed directly by using explicit expressions for v^l_0 and v^l_l, that $v^l_l v^{l*}_0 \sim \eta^l_s$. From the embedding (2.17), we deduce that η^l_s has polar decomposition $\eta^l_s = (S^l, S^l)|\eta^l_s|$. Therefore we can write $v^l_l v^{l*}_0 = (S^l, S^l)|v^l_l v^{l*}_0|$ which implies

$$\sigma(v^l_l v^{l*}_0) = \sigma(|v^l_l v^{l*}_0|)\, U^l.$$

By comparing with (3.22), we see that it now suffices to verify

$$\sigma(v^l_l v^{l*}_l)^{-1/2}\, \sigma(|v^l_l v^{l*}_0|)\, \sigma(v^l_0 v^{l*}_0)^{-1/2} = 1. \tag{3.23}$$

Multiplying both sides of Equation (3.21) with $\sigma(v^l_l v^{l*}_l)$ gives

$$\sigma(v^l_0 v^{l*}_0)\, \sigma(v^l_l v^{l*}_l) = \sigma(v^l_0 v^{l*}_l)\, \sigma(v^l_l v^{l*}_0).$$

Thus

$$\sigma(|v^l_l v^{l*}_0|) = \sigma\big((v^l_0 v^{l*}_l v^l_l v^{l*}_0)^{1/2}\big) = \big(\sigma(v^l_0 v^{l*}_l)\sigma(v^l_l v^{l*}_0)\big)^{1/2} = \big(\sigma(v^l_0 v^{l*}_0)\sigma(v^l_l v^{l*}_l)\big)^{1/2},$$

which proves (3.23). This concludes the proof of the commutativity of (3.18) for $l \geq 0$. The case $l < 0$ is treated analogously.

The surjectivity of $\Psi_- \circ \mathrm{pr}_-$ and $\Psi_+ \circ \mathrm{pr}_+$ follows from the bijectivity of $\Psi_\pm$ and the surjectivity of $\mathrm{pr}_\pm$.

Suppose that $\sum_{k=1}^n (r_k, s_k) \otimes_B a_k \in \ker(\Psi_- \circ \mathrm{pr}_-) \cap \ker(\Psi_+ \circ \mathrm{pr}_+)$. Since $\Psi_\pm$ is an isomorphism, we get $\sum_{k=1}^n r_k \otimes_B a_k = 0$ and $\sum_{k=1}^n s_k \otimes_B a_k = 0$ by (3.17). Hence $\sum_{k=1}^n (r_k, s_k) \otimes_B a_k = \sum_{k=1}^n (r_k, 0) \otimes_B a_k + \sum_{k=1}^n (0, s_k) \otimes_B a_k = 0$ which proves last claim of the lemma. □

We are now in a position to prove the main theorem of this section.

Theorem 3.8. *There is an isomorphism of left $\mathcal{C}(\mathrm{S}^2_q)$-modules and right $\mathcal{O}(\mathrm{U}(1))$-comodules between the fibre product $\mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \times_{(\Phi\circ\pi_1,\pi_2)} \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1))$ from* (3.10) *and $\mathcal{C}(\mathrm{S}^2_q) \otimes_{\mathcal{O}(\mathrm{S}^2_{qs})} \mathcal{O}(\mathrm{SU}_q(2))$. Moreover, the chain of isomorphisms in* (3.11) *holds.*

Proof. Lemma 3.7 states that $\mathcal{C}(\mathrm{S}^2_q) \otimes_{\mathcal{O}(\mathrm{S}^2_{qs})} \mathcal{O}(\mathrm{SU}_q(2))$ is a universal object of the pull back diagram (3.18). Comparing (3.18) and (3.10) shows that both pullback diagrams define up to isomorphism the same universal object which proves the first part of the theorem.

The first isomorphism in (3.11) follows from the Murray-von Neumann equivalence of the corresponding projections, see [17]. The second isomorphism follows from the above equivalence of pullback diagrams, and the last one from fact that all mappings in (3.18) are right $\mathcal{O}(\mathrm{U}(1))$-colinear. □

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